DIFFERENTIAL GEOMETRY OF CURVES IN LAGRANGE GRASSMANNIANS WITH GIVEN YOUNG DIAGRAM

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Abstract. Curves in Lagrange Grassmannians appear naturally in the intrinsic study of geometric structures on manifolds. By a smooth geometric structure on a manifold we mean any submanifold of its tangent bundle, transversal to the fibers. One can consider the time-optimal problem naturally associate with a geometric structure. The Pontryagin extremals of this optimal problem are integral curves of certain Hamiltonian system in the cotangent bundle. The dynamics of the fibers of the cotangent bundle w.r.t. this system along an extremal is described by certain curve in a Lagrange Grassmannian, called Jacobi curve of the extremal. Any symplectic invariant of the Jacobi curves produces the invariant of the original geometric structure. The basic characteristic of a curve in a Lagrange Grassmannian is its Young diagram. The number of boxes in its kth column is equal to the rank of the kth derivative of the curve (which is an appropriately defined linear mapping) at a generic point. We will describe the construction of the complete system of symplectic invariants for parameterized curves in a Lagrange Grassmannian with given Young diagram. It allows to develop in a unified way local differential geometry of very wide classes of geometric structures on manifolds, including both classical geometric structures such as Riemannian and Finslerian structures and less classical ones such as sub-Riemannian and sub-Finslerian structures, defined on nonholonomic distributions.

1. Introduction

Let W be a $2m$-dimensional linear space endowed with a symplectic form ω. Recall that an m-dimensional subspace Λ of W is called Lagrangian, if $\omega|_\Lambda = 0$. Lagrange Grassmannian $L(W)$ of W is the set of all Lagrangian subspaces of W. The linear Symplectic group acts naturally on $L(W)$. Invariants of curves in a Lagrange Grassmannian w.r.t. this action are called symplectic. The present paper is devoted to the construction of a complete system of symplectic invariants for smooth parameterized curves in the Lagrange Grassmannian $L(W)$, i.e., a set of invariants (independent one of each other) such that there exists the unique, up to a symplectic transformation, curve in $L(W)$ with the prescribed invariants from this set. Of course, this problem is a particular case of the classical problem on differential geometry of curves in homogeneous spaces. The general procedure for the latter problem was developed already by E. Cartan with his method of moving frames. On the other hand, by studying curves in Lagrange Grasmannians, one can develop in a unified way local differential geometry of very wide classes of geometric structures on manifolds, including both classical geometric structures such as Riemannian and Finslerian structures and less classical such as sub-Riemannian or sub-Finslerian structures.[1]. Therefore, the explicit construction of moving frames and invariants in the particular situation of curves in Lagrange Grassmannians is important by itself.

Let us briefly describe how curves in Lagrange Grassmannians appear in intrinsic study of geometric structures (more detailed and general presentation can be found in [1] or [2]). Here by a smooth geometric structure on a manifold M we mean any submanifold $\mathcal{V} \subset TM$, transversal

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[1]Differential geometry of rank 2 vector distributions (without additional structures on them) can be treated as well by studying unparameterized curves in Lagrange Grassmannians ([5],[6])

to fibers. Let $\mathcal{V}_q = \mathcal{V} \cap T_qM$. For example, if $\mathcal{V}_q$ is an intersection of an ellipsoid centered at the origin with a linear subspace $\mathcal{D}_q$ in T_qM (where both the ellipsoids and the subspaces $\mathcal{D}_q$ depend smoothly on q), then $\mathcal{V}$ is called a *sub-Riemannian structure on M with underlying distribution* $\mathcal{D}$. In this case $\mathcal{V}_q$ is the unit sphere w.r.t. the unique Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$, i.e. fixing an ellipsoid in $\mathcal{D}_q$ is equivalent to fixing an Euclidean norm on $\mathcal{D}_q$ for any $q \in M$. This reformulation justifies the term "sub-Riemannian". In particular, it defines in the obvious way the length of any curve tangent to the underlying distribution. If in the constructions above we replace the ellipsoids by the boundaries of strongly convex bodies in T_qM (sometimes also assumed to be symmetric w.r.t. the origin) we will get a sub-Finslerian structure on M. Note also that, if the underlying distribution $\mathcal{D} = TM$, we get just a Riemannian (a Finslerian) structure on M.

Actually, one can look at a geometric structure $\mathcal{V}$ as a control system on M: the set $\mathcal{V}_q$ defines the set of all admissible velocities of motion from the point q. A Lipshitzian curve $\gamma : [0,T] \mapsto M$ is called an *admissible trajectory of* $\mathcal{V}$, if $\dot\gamma(t) \in \mathcal{V}_{\gamma}(t)$ for a. e. t. Now one can consider the time-optimal problem on $\mathcal{V}$: given two points q_0 and q_1 to find an admissible trajectory, steering from q_0 to q_1 in a minimal time. The extremals of this optimal problem are obtained from the Pontryagin Maximum Principle of Optimal Control Theory ([8]). Here for simplicity of presentation let us suppose that the maximized Hamiltonian of the Pontryagin Maximum Principle

$$H(p,q) = \max_{v \in \mathcal{V}_q} p(v), \quad q \in M, p \in T_q^*M \tag{1.1}$$

is well defined and smooth in an open domain $O \subset T^*M$ and for some $c > 0$ (and therefore for any $c > 0$ by homogeneity of H on each fiber of T^*M) the corresponding level set

$$\mathcal{H}_c = \{\lambda \in O : H(\lambda) = c\}$$

is nonempty and consists of regular points of H. Consider the Hamiltonian vector field $\vec{H}$ on $\mathcal{H}_c$, corresponding to the Hamiltonian H, i.e. the vector field satisfying $i_{\vec{H}}\bar\omega = -dH$, where $\bar\omega$ is the canonical symplectic structure on T^*M. The integral curves of this Hamiltonian system are normal Pontryagin extremals of the time-optimal problem, associated with geometric structure $\mathcal{V}$, or, shortly, normal extremals of $\mathcal{V}$. For example, if $\mathcal{V}$ is a sub-Riemannian structure with underlying distribution $\mathcal{D}$, then the maximized Hamiltonian satisfies $H(p,q) = ||p|_{\mathcal{D}_q}||_q$, i.e. $H(p,q)$ is equal to the norm of the restriction of the functional $p \in T_q^*M$ on $\mathcal{D}_q$ w.r.t. the Euclidean norm $||\cdot||_q$ on $\mathcal{D}_q$; $O = T^*M \backslash \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ is the annihilator of $\mathcal{D}$,

$$\mathcal{D}^\perp = \{(p,q) \in T^*M : p(v) = 0 \ \forall v \in \mathcal{D}_q\}.$$

The projections of the trajectories of the corresponding Hamiltonian systems to the base manifold M are normal sub-Riemannian geodesics. If $\mathcal{D} = TM$, then they are exactly the Riemannian geodesics of the corresponding Riemannian structure.

Further let $\mathcal{H}_c(q) = \mathcal{H}_c \cap T_q^*M$. $\mathcal{H}_c(q)$ is a codimension 1 submanifold of T_q^*M. For any $\lambda \in \mathcal{H}_c$ denote $\Pi_\lambda = T_\lambda\big(\mathcal{H}_c(\pi(\lambda))\big)$, where $\pi : T^*M \mapsto M$ is the canonical projection. Actually Π_λ is the vertical subspace of $T_\lambda\mathcal{H}_c$,

$$\Pi_\lambda = \{\xi \in T_\lambda\mathcal{H}_c : \pi_*(\xi) = 0\}. \tag{1.2}$$

Now with any integral curve of $\vec{H}$ one can associate a curve in a Lagrange Grassmannian, which describes the dynamics of the vertical subspaces Π_λ along this integral curve w.r.t. the flow $e^{t\vec{H}}$, generated by $\vec{H}$. For this let

$$t \mapsto J_\lambda(t) \stackrel{def}{=} e_*^{-t\vec{H}}\big(\Pi_{e^{t\vec{H}}\lambda}\big)\Big/\{\mathbb{R}\vec{H}(\lambda)\}. \tag{1.3}$$

The curve $J_\lambda(t)$ is the curve in the Lagrange Grassmannian of the linear symplectic space $W_\lambda = T_\lambda\mathcal{H}_c/\{\mathbb{R}\vec{H}(\lambda)\}$ (endowed with the symplectic form ω induced in the obvious way by the canonical symplectic form $\bar{\omega}$ of T^*M). It is called *the Jacobi curve of the curve* $e^{t\vec{\mathcal{H}}}\lambda$ *attached at the point* λ. Note also that if $\bar{\lambda} = e^{\bar{t}\vec{H}}\lambda$ and $\Phi : W_\lambda \mapsto W_{\bar{\lambda}}$ is a symplectic transformation induced in the natural way by a linear mapping $e_*^{\bar{t}\vec{H}} : T_\lambda\mathcal{H}_c \mapsto T_{\bar{\lambda}}\mathcal{H}_c$, then by (1.3) we have

$$J_{\bar{\lambda}}(t) = \Phi\big(J_\lambda(t - \bar{t})\big). \tag{1.4}$$

In other words, the Jacobi curves of the same integral curve of $\vec{H}$ attached at two different points of this curve are the same, up to symplectic transformation between the corresponding ambient linear symplectic spaces and the corresponding shift of the parameterizations. Therefore, any symplectic invariant of the Jacobi curve produces the function on the manifold $\mathcal{H}_c$, intrinsically related to the geometric structure V (the value of this function at $\lambda \in \mathcal{H}_c$ is equal to the value of the chosen symplectic invariant of the curve $J_\lambda(t)$ at $t = 0$). In this way the problem of finding differential invariants of geometric structure can be essentially reduced to the much more treatable problem of finding symplectic invariants of certain curves in a Lagrange Grassmannian.

In all constructions above one can replace the maximized Hamiltonian H by some its power H^s. It causes only the reparametrization of the Jacobi curve of the type $t \mapsto Ct$ for some constant C. For example in the case of sub-Riemannian structures it is more convenient to work with H^2 instead of H, because H^2 is a polynomial on the fibers of T^*M.

Jacobi curves of integral curves of $\vec{H}$ are not arbitrary curves of Lagrangian Grassmannian but they inherit special features of the geometric structure $\mathcal{V}$. To specify these features recall that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian $L(W)$ at the point Λ can be naturally identified with the space $\mathrm{Quad}(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$. Namely, given $\mathfrak{V} \in T_\Lambda L(W)$ take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$ and $\dot{\Lambda} = \mathfrak{V}$. Given some vector $l \in \Lambda$, take a curve $\ell(\cdot)$ in W such that $\ell(t) \in \Lambda(t)$ for all t and $\ell(0) = l$. Define the quadratic form

$$Q_{\mathfrak{V}}(l) = \omega(\frac{d}{dt}\ell(0), l). \tag{1.5}$$

Using the fact that the spaces $\Lambda(t)$ are Lagrangian, it is easy to see that $Q_{\mathfrak{V}}(l)$ does not depend on the choice of the curves ℓ and $\Lambda(t)$ with the above properties, but depends only on $\mathfrak{V}$. So, we have the linear mapping from $T_\Lambda L(W)$ to the spaces $\mathrm{Quad}(\Lambda)$, $\mathfrak{V} \mapsto Q_{\mathfrak{V}}$. A simple counting of dimensions shows that this mapping is a bijection and it defines the required identification. A curve $\Lambda(\cdot)$ in a Lagrange Grassmannian is called *regular at a point* τ, if its velocity at τ is a nondegenerated quadratic form, and *nonregular at* τ otherwise. The rank of the velocity $\dot{\Lambda}(\tau)$ of a curve $\Lambda(\cdot)$ at a point τ is called shortly the *rank of* $\Lambda(\cdot)$ at τ. A curve $\Lambda(\cdot)$ is called *monotonically nondecreasing (nonincreasing)* if the velocity is nonnegative (nonpositive) definite at any point. We also will call such curves *monotonic.*

It turns out (see, for example, [2, Proposition 1]) that the velocity of the Jacobi curve $J_\lambda(t)$ at $t = 0$ is equal to the restriction of the Hessian of H to the tangent space to $\mathcal{H}_{H(\lambda)}$ at λ. This together with (1.4) implies easily ([2]) that the rank of the Jacobi curve $J_\lambda(t)$ at $t = \tau$ is not greater then $\dim \mathcal{V}_{\pi(e^{\tau\vec{H}}\lambda)}$. For sub-Riemannian structures the rank of Jacobi curves at any point is equal to $\operatorname{rank}\mathcal{D} - 1$, where $\mathcal{D}$ is the underlying distribution, i.e., except the case $\mathcal{D} = TM$ (corresponding to a Riemannian structure), the Jacobi curves appearing in sub-Riemannian structures are nonregular at any point.

Regular curves were treated in [1], where the notion of the curvature operator was introduced (the work [7] is closely related as well). In particular, calculating the curvature operator for Jacobi curves, associated with a Riemannian structure, one gets a part of the Riemannian curvature tensor, appearing in the classical Jacobi equation for Jacobi vector fields along the Riemannian geodesics. The whole Riemannian curvature tensor can be recovered uniquely from it.

Basic symplectic invariants of curves (both parameterized and unparameterized) in Lagrange Grassmannians, which are nonregular at any point, were constructed in [2], using the notion of cross-ratio of four points in Lagrange Grassmannians. But the only nonregular (at any point) curves in Lagrange Grassmannians, for which the complete system of symplectic invariants was constructed, were parameterized curves of constant rank 1 ([9]).

In the present paper we develop differential geometry of curves of any constant rank in Lagrange Grassmannians, implementing the scheme briefly described in the Introduction of [9]. In the study of generic germs of nonregular curves the basic characteristic are not only the rank of its velocity, but a certain Young diagram (see subsections 2.1). The rank of the curve is the number of boxes in the first column of this Young diagram. It is also very convenient to consider the additional "smaller ' diagram, called the reduced Young diagram (see subsections 2.2). For a regular curve the Young diagram consists of one column and the reduced Young diagram consists of one box, while for rank 1 curve the Young diagram and its reduction coincide and consist of one row. For any monotonic curve or a generic nonmonotonic curve $\Lambda(\cdot)$ in a Lagrange Grassmannian with given Young diagram we construct the principal bundle (over the curve itself) of frames in the ambient symplectic space endowed with the canonical principal connection or the bundle of moving frames, canonically associated with the curve (Theorems 1 and 3). These moving frames are defined by the form of the matrix in their structural equation. During the process of normalization we get the canonical splitting of any subspaces $\Lambda(t)$ such that the subspaces of the splitting are parameterized by boxes of the reduced Young diagram and each subspace of the splitting is endowed with the canonical Euclidean or pseudo-Euclidean structure (in the monotonic and nonmonotonic cases repsectively). Also we construct in a canonical way the additional curve $\Lambda^{trans}(\cdot)$ in a Lagrange Grassmannian such that any subspace $\Lambda^{trans}(t)$ is transversal to the subspace $\Lambda(t)$ for any t. Further, using the matrix in the structural equation of canonical moving frames, we obtain the tuple of one-parametic families of linear mappings between the subspaces of the canonical splitting. This tuple constitutes a kind of a complete system of symplectic invariants of the curve in a sense formulated in terms of quivers and their representations (Theorems 2 and 4). In the case when the Young diagram of the curve $\Lambda(\cdot)$ has no rows with the same number of boxes, we get in this way a complete system of scalar invariants of the curve $\Lambda(\cdot)$ in the usual sense. As the consequences of our constructions in section 5 we get the canonical (non-linear) connection on an open sets of the cotangent bundle T^*M, the curvature-type invariants, and additional nontrivial structures on the fibers of T^*M for geometric structures on a manifold M, including sub-Riemannian and sub-Finslerian structures, satisfying very general assumptions.

2. The main results

2.1. **The flag and the Young diagrams associated with a curve.** With any curve $\Lambda(\cdot)$ in Grassmannian $G_k(W)$ of k-dimensional subspaces of a linear space W one can associate a curve of flags of subspaces in W. For this let $\mathfrak{S}(\Lambda)$ be the set of all smooth curves $\ell(t)$ in W such that $\ell(t) \in \Lambda(t)$ for all t. Denote

$$\Lambda^{(i)}(\tau) = \operatorname{span}\Big\{\frac{d^j}{d\tau^j}\ell(\tau)| : \ell \in \mathfrak{S}(\Lambda),\, 0 \le j \le i\Big\}. \tag{2.1}$$

The subspaces $\Lambda^{(i)}(\tau)$ are called *the ith extension* of the curve $\Lambda(\cdot)$ at the point τ. Recall that the tangent space $T_\Lambda G_k(W)$ to any subspace $\Lambda \in G_k(W)$ can be identified with the space $\operatorname{Hom}(\Lambda, W/\Lambda)$ of linear mappings from Λ to W/Λ. Using this identification, if $P : \Lambda \mapsto W/\Lambda$ is the canonical projection to the factor, then $\Lambda^{(1)}(\tau) = P^{(-1)}\big(\operatorname{Im}\dot\Lambda(\tau)\big)$, which implies that $\dim \Lambda^{(1)}(\tau) - \dim \Lambda(\tau) = \operatorname{rank} \dot\Lambda(\tau)$. By construction $\Lambda^{(i-1)}(\tau) \subseteq \Lambda^{(i)}(\tau)$. The flag

$$\Lambda(\tau) \subseteq \Lambda^{(1)}(\tau) \subseteq \Lambda^{(2)}(\tau) \subseteq \dots \tag{2.2}$$

is called the *associated (right) flag of the curve* $\Lambda(\cdot)$ *at the point* t.

From now on we suppose that dimensions of all subspaces $\Lambda^{(i)}(t)$ (and therefore of $\Lambda_{(i)}(t)$) are independent of t. In this case from (2.1) it is easy to obtain that the following inequalities hold

$$\dim \Lambda^{(i+1)} - \dim \Lambda^{(i)} \leq \dim \Lambda^{(i)} - \dim \Lambda^{(i-1)}. \tag{2.3}$$

Using inequalities (2.3), to any curve $\Lambda(\cdot)$ we can assign the Young diagram in the following way: the number of boxes in the ith column of this Young diagram is equal to $\dim \Lambda^{(i)} - \dim \Lambda^{(i-1)}$. It will be called the *Young diagram of the curve* $\Lambda(\cdot)$. In particular, the number of boxes in the first column is equal to the rank of the curve.

Now suppose that W is an even-dimensional linear space endowed with a symplectic structure ω and the curve $\Lambda(\cdot)$ is a curve in the Lagrangian Grassmannian $L(W)$.

Remark 1. Without loss of generality, we will suppose that there exists an integer p such that $\Lambda^{(p)}(t) = W$. Otherwise, if $\Lambda^{(p+1)}(t) = \Lambda^{(p)}(t) \subsetneq W$, then the subspace $\Lambda^{(p)}(t)$ does not depend on t. Set $V = \Lambda^{(p)}(t)$. Then $V^{\angle} \subset \Lambda(t)$ for any t and all information about the original curve $\Lambda(\cdot)$ is contained in the curve $\Lambda(\cdot)/V^{\angle}$, which is the curve of Lagrangian subspaces in the symplectic space $V/V^{\angle}$, and the pth extension of the curve $\Lambda(\cdot)/V^{\angle}$ is equal to $V/V^{\angle}$. So, we can work with the curve $\Lambda(\cdot)/V^{\angle}$ and the symplectic space $V/V^{\angle}$ instead of the curve $\Lambda(\cdot)$ and the symplectic space W.

2.2. **The normal moving frame.** The Young diagram is a basic invariant of the curve in Lagrange Grassmannians. As indices of vectors in our Darboux moving frames we will take the boxes of the Young diagram instead of the natural numbers. We found it extremely useful both for formulation of our results and their proofs.

First note that any Young diagram D can be uniquely represented as a union of d rectangular diagrams D_i of the sizes $r_i \times p_i$, $1 \leq i \leq d$, such that the sequence $\{p_i\}_{i=1}^d$ is strictly decreasing. The Young diagram Δ, consisting of d rows such that the ith row has p_i boxes, will be called the *r*educed diagram or the *reduction of the diagram D.* In order to distinguish between boxes and rows of the diagram D and its reduction Δ, the boxes of Δ will be called *superboxes* and the rows of Δ will be called *levels.* To the jth superbox a of the ith level of Δ one can assign the jth column of the rectangular subdiagram D_i of D and the integer number r_i (equal to the number of boxes in this subcolumn), called the *size* of the superbox a.

As usual, by $\Delta \times \Delta$ we will mean the set of pairs of superboxes of Δ. Also denote by Mat the set of matrices of all sizes. The mapping $R : \Delta \times \Delta \mapsto \text{Mat}$ is called *compatible with the Young diagram D*, if to any pair (a, b) of superboxes of sizes s_1 and s_2 respectively the matrix $R(a, b)$ is of the size $s_2 \times s_1$. The compatible mapping R is called *symmetric* if for any pair (a, b) of superboxes the following identity holds

$$R(b, a) = R(a, b)^T. \tag{2.4}$$

Denote by Υ_i the ith level of Δ. Also denote by a_i and σ_i the first and the last superboxes of the ith level Υ_i respectively and by $r : \Delta\backslash\{\sigma_i\}_{i=1}^d \mapsto \Delta$ the right shift on the diagram Δ. The last superbox of any level will be called *special.* For any pair of integers (i, j) such that $1 \leq j < i \leq d$ consider the following tuple of pairs of superboxes

$$\begin{aligned}&\big(a_j, a_i\big),\ \big(a_j, r(a_i)\big),\ \big(r(a_j), r(a_i)\big),\ \big(r(a_j), r^2(a_i)\big), \ldots, \big(r^{p_i-1}(a_j), r^{p_i-1}(a_i)\big),\\ &\big(r^{p_i}(a_j), r^{p_i-1}(a_i)\big), \ldots, \big(r^{p_j-1}(a_j), r^{p_i-1}(a_i)\big).\end{aligned} \tag{2.5}$$

Actually the tuple (2.5) is obtained as follows: the first pair consists of the last two superboxes of the considered levels, then until the superbox of the ith level will not become special, each next even pair is obtained from the previous pair of the tuple by the right shift of the superbox of the ith level in the previous pair and each next odd pair is obtained from the previous pair of the tuple by the right shift of the superbox of the jth level in the previous pair. When the superbox of the ith level become special, each next pair is obtained from the previous pair of the tuple by the right shift of the superbox of the jth level.

Now we are ready to introduce two crucial notions, which will be very useful in the formulation of our main Theorem:

Definition 1. *A symmetric compatible mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ is called quasi-normal if the following two conditions hold:*

1. *Among all matrices $\mathcal{R}(a,b)$, where the superbox b is not higher than the superbox a in the diagram Δ, the only possible nonzero matrices are the following: the matrices $\mathcal{R}(a,a)$ for all $a \in \Delta$, the matrices $R\big(a, r(a)\big)$, $R\big(r(a), a\big)$ for all nonspecial boxes, and the matrices, corresponding to the pairs, which appear in the tuples (2.5), for all $1 \le j < i \le d$;*
2. *The matrix $R\big(a, r(a)\big)$ is antisymmetric for any nonspecial superbox a.*

Definition 2. *A quasi-normal mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ is called normal if it satisfies the following condition: for any $1 \le j < i \le d$, the matrices, corresponding to the first $(p_j - p_i - 1)$ pairs of the tuple (2.5), are equal to zero.*

Now let us fix some terminology about the frames in W, indexed by the boxes of the Young diagram D. A frame $\big(\{e_\alpha\}_{\alpha \in D}, \{f_\alpha\}_{\alpha \in D}\big)$ of W is called *Darboux* or *symplectic*, if for any $\alpha, \beta \in D$ the following relations hold

$$\omega(e_\alpha, e_\beta) = \omega(f_\alpha, f_\beta) = \omega(f_\alpha, e_\beta) - \delta_{\alpha,\beta} = 0, \tag{2.6}$$

where $\delta_{\alpha,\beta}$ is the analogue of the Kronecker index defined on $D \times D$. In the sequel it will be convenient to divide a moving frame $\big(\{e_\alpha(t)\}_{\alpha \in D}, \{f_\alpha(t)\}_{\alpha \in D}\big)$ of W indexed by the boxes of the Young diagram D into the tuples of vectors indexed by the supeboxes of the reduction Δ of D, according to the correspondence between the superboxes of Δ and the subcolumns of D. More precisely, given a superbox a in Δ of size s, take all boxes $\alpha_1, \dots, \alpha_s$ of the corresponding subcolumn in D in the order from the top to the bottom and denote

$$E_a(t) = \big(e_{\alpha_1}(t), \dots, e_{\alpha_s}(t)\big), \quad F_a(t) = \big(f_{\alpha_1}(t), \dots, f_{\alpha_s}(t)\big).$$

In what follows we will suppose that the curve $\Lambda(t)$ is monotonically nondecreasing, i.e. the velocity $\dot{\Lambda}(t)$ is a nonnegative definite quadratic form for any t. The case of monotonically nonincreasing curve can be treated then by reversing of time. We restrict ourselves to the monotonic curves just in order to avoid technicalities both in the formulation and the proof of our main result (Theorem 1 below). The similar result with essentially the same proof is valid also for nonmonotonic curves under additional generic assumptions, which will be introduced in subsection 3.3 (see Condition (G) there). In section 4 we point out what changes one has to make in Theorem 1 in nonmonotonic situation (see Theorem 3 below). Note also that Jacobi curves in sub-Riemannian and, more generally, in sub-Finslerian geometry are monotonic, because the corresponding maximized Hamiltonians are convex on the fibers of T^*M (see the Introduction).

Definition 3. *The moving Darboux frame $(\{E_a(t)\}_{a \in \Delta}, \{F_a(t)\}_{a \in \Delta})$ is called the normal (quasi-normal) moving frame of a monotonically nondecreasing curve $\Lambda(t)$ with the Young diagram D if $\Lambda(t) = \mathrm{span}\{E_a(t)\}_{a \in \Delta}$ for any t and there exists an one-parametric family of normal (quasi-normal) mappings $R_t : \Delta \times \Delta \mapsto \mathrm{Mat}$ such that the moving frame $(\{E_a(t)\}_{a \in \Delta}, \{F_a(t)\}_{a \in \Delta})$ satisfies the following structural equation:*

$$\begin{cases} E_a'(t) = E_{l(a)}(t) & \text{if } a \in \Delta \setminus \mathcal{F}_1 \\ E_a'(t) = F_a(t) & \text{if } a \in \mathcal{F}_1 \\ F_a'(t) = \sum\limits_{b \in \Delta} E_b R_t(a,b) - F_{r(a)} & \text{if } a \in \Delta \setminus \mathcal{S} \\ F_a'(t) = \sum\limits_{b \in \Delta} E_b R_t(a,b) & \text{if } a \in \mathcal{S} \end{cases}, \tag{2.7}$$

where $\mathcal{F}_1$ is the first column of the diagram Δ, $\mathcal{S}$ is the set of all its special superboxes, and $l : \Delta\backslash\mathcal{F}_1 \mapsto \Delta$, $r : \Delta\backslash\, \mathcal{S} \mapsto \Delta$ are the left and right shifts on the diagram Δ. The mapping R_t, appearing in (2.7), is called the normal (quasi-normal) mapping, associated with the normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$.

With all this terminology we are ready to formulate our main theorem:

Theorem 1. *For any monotonically nondecreasing curve $\Lambda(t)$ with the Young diagram D in the Lagrange Grassmannian there exists a normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$. A moving frame $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ is a normal moving frame of the curve $\Lambda(\cdot)$ if and only if for any $1 \le i \le d$ there exists a constant orthogonal matrix U_i of size $r_i \times r_i$ such that for all t*

$$\widetilde{E}_a(t) = E_a(t)U_i, \quad \widetilde{F}_a(t) = F_a(t)U_i, \quad \forall\, a \in \Upsilon_i. \tag{2.8}$$

Actually, the second statement of this theorem means that if for any $\bar{t}$ one collects all possible Darboux frame $(\{E_a\}_{a\in\Delta}, \{F_a)\}_{a\in\Delta})$ in W such that there exists a normal moving frame, which coincides with $(\{E_a\}_{a\in\Delta}, \{F_a)\}_{a\in\Delta})$ at $t = \bar{t}$, then one gets the principle $O(r_1) \times \ldots \times O(r_d)$ bundle over the curve $\Lambda(t)$ endowed with the canonical principal connection in the following way: the normal moving frames are horizontal curves w.r.t. this connection.

2.3. **The canonical splitting and curvature operators.** Before proving Theorem 1 let us discuss it a little bit. Take some normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$. Relations (2.8) imply that for any superbox $a \in \Delta$ of size s the following s-dimensional subspace

$$V_a(t) = \mathrm{span}\{E_a(t)\} \tag{2.9}$$

of $\Lambda(t)$ does not depend on the choice of the normal moving frame. The subspace V_a will be called the *subspace, associated with the superbox a*. So, there exists the *canonical splitting of the subspace* $\Lambda(t)$:

$$\Lambda(t) = \bigoplus_{a\in\Delta} V_a(t). \tag{2.10}$$

Moreover, each subspace $V_a(t)$ is endowed with the *canonical Euclidean structure* such that the tuple of vectors E_a constitute an orthonormal frame w.r.t. to it. Note that the canonical splitting is obtained in one of the first steps of the normalization procedure in the proof of Theorem 1 (see section 3.4)

Another very important consequence of (2.8) is that the following subspace

$$\Lambda^{\mathrm{trans}}(t) = \bigoplus_{a\in\Delta} \mathrm{span}\{F_a(t)\} \tag{2.11}$$

does not depend on the choice of the normal moving frame. By construction, $W = \Lambda(t) \oplus \Lambda^{\mathrm{trans}}(t)$ for any t. The curve $\Lambda^{\mathrm{trans}}(t)$ will be called the *canonical complementary curve of the curve* $\Lambda(\cdot)$. As we will see in section 5 this notion is crucial for the construction of the canonical (non-linear) connection for sub-Riemannian and, more generally, sub-Finsler structures. [2]

Further, we say that a pair (a, b) of superboxes is *essential* if $R(a, b)$ is not necessarily zero for a normal mapping $R : \Delta \times \Delta \mapsto \mathrm{Mat}$. Note that this notion depends only on the mutual

[2]Note also that this curve is different in general from the so-called derivative curve $\Lambda^0(\cdot)$, constructed in [2], which is also intrinsically related to $\Lambda(\cdot)$ such that the space $\Lambda^0(t)$ is transversal to $\Lambda(t)$ for any t. The main disadvantage of the derivative curve $\Lambda^0(\cdot)$, comparing to the curve $\Lambda^{\mathrm{trans}}(\cdot)$, constructed here, is that if one uses it for the construction of the moving frames intrinsically related to the curve $\Lambda(\cdot)$, as was done in [2] and [3], then it is very hard to analyze their structural equations and to distinguish a complete system of invariants from it (in the mentioned papers it was partially done only in the case of curves of rank 1), while in the present paper we construct the normal moving frame step by step according to the heuristic rule that the matrix of its structural equation should be as simple as possible (should contain as much zeros as possible), which gives the complete system of invariants automatically.

locations of the superboxes a and b in the diagram Δ, except the case of consecutive superboxes a and b in the same level of Δ. In the last case it depends on the size of the superboxes. Namely, the pair $\big(a, r(a)\big)$ is essential if and only if the size of a is greater than 1 (see condition (1) of Definition 1).

Assume that $R_t : \Delta \times \Delta \mapsto \text{Mat}$ and $\widetilde{R}_t : \Delta \times \Delta \mapsto \text{Mat}$ are the normal mappings, associated with normal moving frames $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ and $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$, which are related by (2.8). Then from (2.7) and (2.8) it follows immediately that

$$\widetilde{R}_t(a,b) = U_j^{-1} R_t(a,b) U_i, \quad a \in \Upsilon_i, b \in \Upsilon_j. \tag{2.12}$$

The last relation means actually that for any essential pair (a,b) of superboxes the linear mapping $\mathfrak{R}_t(a,b) : V_a \mapsto V_b$, having the matrix $R_t(a,b)$ w.r.t. the bases E_a and E_b of V_a and V_b respectively, does not depend on the choice of a normal moving frame.[3] The linear mapping $\mathfrak{R}_t(a,b)$ will be called the *(a,b)-curvature mapping of the curve* $\Lambda(\cdot)$.

The only nontrivial blocks in the matrix of the structural equations for the normal moving frames correspond to (a,b)-curvature mappings. Hence the tuple of all (a,b)-curvature mappings constitute a kind of complete system of symplectic invariants of the curve. For precise formulation of this statement it is convenient to use the notion of quivers and their representations ([4]). Recall that a quiver is an oriented graph, where loops and multiple arrows between two vertices are allowed. A representation of a quiver assigns a vector space X_α to each vertex α of the quiver and a linear mapping from X_α to X_β to each arrow of the quiver, connecting a vertex α with a vertex β.

Take the quiver $\mathfrak{Q}_D$ such that its vertices are levels of the diagram Δ and the set of arrows from the level Υ_i to the level Υ_j is parameterized by essential pairs $(a,b) \in \Upsilon_i \times \Upsilon_j$. A representation of the quiver $\mathfrak{Q}_D$ will be called *compatible with the Young diagram D* if for any $1 \le i \le d$ the space of the representation corresponding to the vertex Υ_i is a r_i-dimensional Euclidean space and the linear mappings $\mathcal{R}(a,b)$ of the representation corresponding to the arrows (a,b) satisfy the following relations: $\mathcal{R}(a,b)^* = \mathcal{R}(b,a)$ and $\mathcal{R}\big(a, r(a)\big)$ are antisymmetric w.r.t. the corresponding Euclidean structure.

The subspaces $V_a(t)$ for any t and any $a \in \Upsilon_i$ are naturally identified together with the canonical Euclidean structure on them ($V_{a_1}(t_1) \sim V_{a_2}(t_2)$ by sending $E_{a_1}(t_1)$ to $E_{a_2}(t_2)$). Therefore, we can identify all these spaces with one Euclidean space, which will be denoted by $\mathcal{X}_i$. The tuple of spaces $\mathcal{X}_i$ and the (a,b)-curvatures mappings of the curve $\Lambda(t)$, considered as elements of $\text{Hom}(\mathcal{X}_i, \mathcal{X}_j)$ for $(a,b) \in \Upsilon_i \times \Upsilon_j$, define the one-parametric family $\mathfrak{R}_t$ of compatible representations of the quiver $\mathfrak{Q}_D$. This family will be called the *quiver of curvatures of the curve* $\Lambda(t)$. Here the linear mappings corresponding to the arrows of the quiver depend on t, while the linear spaces, corresponding to its vertices, are independent of t. In the sequel we will consider only this type of one-parametric families of representations of quivers. Two families $\Xi_1(t)$ and $\Xi_2(t)$ of compatible representations of the quiver $\mathfrak{Q}_D$ are called isomorphic, if there exists a tuple of isometries (independent of t) between the corresponding spaces of the representations, conjugating all corresponding linear mappings. If the sizes of all superboxes in Δ are equal to 1, then the normal moving frames of the curve are defined up to the discrete group (U_i in (2.8) are scalars, which are equal to 1 or -1) and all (a,b)-curvature mappings are determined by scalar functions of t, which are symplectic invariants of the curve. These scalar functions will be called, for short, *(a,b)-curvatures*. Besides, the compatible representations of the quiver $\mathfrak{Q}_D$ is in one-to-one correspondence with tuples of numbers parameterized by the essential pairs of Δ (which is equal to D in the considered case). The following theorem is the direct consequence of the structural equations for normal moving frames and Theorem 1:

[3] Here we restrict ourselves to essential pairs, because for nonessential pairs such linear mappings are zeros automatically.

Theorem 2. *For the given one-parametric family $\Xi(t)$ of representations of the quiver $\mathfrak{Q}_D$ compatible with the Young diagram D with $|D|$ boxes there exists the unique, up to a symplectic transformation, monotonically nondecreasing curve $\Lambda(t)$ in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that the quiver of curvatures of $\Lambda(t)$ is isomorphic to $\Xi(t)$. If, in addition, all rows of D have different length, then given a tuple of smooth functions $\{\rho_{a,b}(t) : (a,b) \in \Delta \times \Delta, (a,b)$ is an essential pair$\}$ there exists the unique, up to a symplectic transformation, monotonically nondecreasing curve $\Lambda(t)$ in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that for any essential pair $(a,b) \in \Delta \times \Delta$ and any t its (a,b)-curvature at t coincides with $\rho_{a,b}(t)$.*

Finally note that rank 1 curves in Lagrange Grassmannians, considered in [9], have the Young diagrams, consisting of just one row, and the main results of the mentioned paper (Theorems 2 and 3 there) are very particular cases of Theorems 1 and 2 here. In this case the pair (a,b) of superboxes is essential if and only if $a = b$.

3. Proof of Theorem 1

The proof consists of several steps.

3.1. **Contractions of the curve $\Lambda(\cdot)$.** We start with some general constructions for curves in Grassmannians. Given a curve $\Lambda(\cdot)$ in the Grassmannian $G_k(W)$, for any τ we will construct a monotonic sequence of subspaces of $\Lambda(\tau)$ in addition to the extensions $\Lambda^{(i)}$. For this let $\Lambda_{(0)}(t) = \Lambda(t)$ and recursively

$$\Lambda_{(i)}(\tau) = \left\{ v \in \Lambda_{(i-1)}(\tau) : \begin{array}{l} \exists\, \ell \in \mathfrak{S}(\Lambda_{(i-1)}) \text{ with } \ell(\lambda) = v \\ \text{such that } \ell'(\tau) \in \Lambda_{(i-1)}(\tau) \end{array} \right\} \tag{3.1}$$

where, by analogy with above, $\mathfrak{S}(\Lambda_{(i)})$, $i \geq 0$, is the set of all smooth curves $\ell(t)$ in W such that $\ell(t) \in \Lambda_{(i-1)}(t)$ for any t. The subspaces $\Lambda_{(i)}(\tau)$ are called *the ith contraction* of the curve $\Lambda(\cdot)$ at the point τ. Under the identification $T_\Lambda G_k(W) \sim \mathrm{Hom}\,(\Lambda, W/\Lambda)$ the first contraction $\Lambda_{(1)}(\tau)$ is exactly the kernel of the velocity $\dot{\Lambda}(\tau)$, $\Lambda_{(1)}(\tau) = \mathrm{Ker}\,\dot{\Lambda}(\tau)$. In particular, it implies that

$$\dim \Lambda^{(1)}(\tau) - \dim \Lambda(\tau) = \dim \Lambda(\tau) - \dim \Lambda_{(1)}(\tau). \tag{3.2}$$

Indeed, the righthand side of (3.2) is equal to $\dim\big(\mathrm{Im}\,\dot{\Lambda}(\tau)\big)$, while the lefthand side is equal to $\dim \Lambda(\tau) - \dim\big(\mathrm{Ker}\,\dot{\Lambda}(\tau)\big)$.

Note also that in (3.1) one can replace the quantor $\exists$ by $\forall$, because the existence of a curve $\ell \in \mathfrak{S}(\Lambda_{(i-1)})$ with $\ell(\tau) = v$ and $\ell'(\tau) \in \Lambda_{(i-1)}(\tau)$ implies that any smooth curve $\tilde{\ell} \in \mathfrak{S}(\Lambda_{(i-1)})$ with $\tilde{\ell}(\tau) = v$ satisfies $\tilde{\ell}'(\tau) \in \Lambda_{(i-1)}(\tau)$. Note that the following relations follow directly from the definitions

$$\big(\Lambda_{(i)}(\tau)\big)_{(1)} = \Lambda_{(i+1)}(\tau), \quad \big(\Lambda_{(i)}(\tau)\big)^{(1)} \subseteq \Lambda_{(i-1)}(\tau) \tag{3.3}$$

If we suppose that $\Lambda(\cdot)$ is a curve in Lagrange Grassmannian of the symplectic space W, then the symplectic structure gives an additional relation between the ith extension and the ith contraction. Namely, given a subspace $L \subset W$ denote by $L^\angle$ its skew-symmetric complement, i.e. $L^\angle = \{v \in W : \omega(v,l) = 0 \ \forall l \in L\}$.

Lemma 1. *The subspaces $\Lambda_{(i)}(\tau)$ is a skew-symmetric complement of the subspace $\Lambda^{(i)}(\tau)$ for any τ, namely*

$$\Lambda_{(i)}(\tau) = \left(\Lambda^{(i)}(\tau)\right)^\angle, \quad \forall \tau. \tag{3.4}$$

Proof. We proceed the proof by induction on i. For $i=0$ there is nothing to prove, because $\Lambda(\tau)$ ($=\Lambda^0(\tau)=\Lambda_0(\tau)$ by definition) is a Lagrangian subspace. Assume that (3.4) is valid for $i=\bar{i}-1$ and prove it for $i=\bar{i}$, $\bar{i}\geq 1$. Indeed, if $v\in\Lambda_{(\bar{i})}(\tau)$, then by definition there exists a regular curve of vectors $v(t)$ such that $v(t)\in\Lambda_{(\bar{i}-1)}(t)$ for any t close to τ, $v(\tau)=v$ and $v'(\tau)\in\Lambda_{(\bar{i}-1)}(\tau)$. Let us prove that $v\in\big(\Lambda_{(\bar{i})}(\tau)\big)^{\angle}$. For this take $v_1\in\Lambda^{(\bar{i})}(\tau)$. Then by definition there exist a curve of vectors $w(t)$ in W such that $w(t)\in\Lambda^{(\bar{i}-1)}(t)$ for any t close to τ and $w'(\tau)=v_1$. By induction hypothesis $\omega\big(v(t),w(t)\big)=0$. Differentiating the last identity at $t=\tau$ we get

$$\omega(v,v_1)=-\omega\big(v'(\tau),w(\tau)\big)=0. \tag{3.5}$$

(the last equality holds because of the relations $v'(\tau)\in\Lambda_{(\bar{i}-1)}(\tau)$, $w(\tau)\in\Lambda^{(\bar{i}-1)}(\tau)$ and the induction hypothesis). Since (3.5) holds for any $v_1\in\Lambda^{(\bar{i})}(\tau)$, we get that $v\in\big(\Lambda_{(\bar{i})}(\tau)\big)^{\angle}$. So, we have proved that $\Lambda_{(i)}(\tau)\subset\Big(\Lambda^{(i)}(\tau)\Big)^{\angle}$.

Now let us prove the inclusion in the opposite direction. Suppose that $v\in\Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle}$. Take any $w\in\Lambda^{(\bar{i}-1)}(\tau)$ and a curve of vectors $w(t)$ in W such that $w(t)\in\Lambda^{(\bar{i}-1)}(t)$ for any t close to τ and $w(\tau)=w$. Then by definition $w'(\tau)\in\Lambda^{(\bar{i})}(\tau)$ and by our assumptions

$$\omega\big(v,w'(\tau)\big)=0. \tag{3.6}$$

On the other hand, since $\Lambda^{(\bar{i}-1)}(\tau)\subset\Lambda^{(\bar{i})}(\tau)$, then $\Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle}\subset\Big(\Lambda^{(\bar{i}-1)}(\tau)\Big)^{\angle}=\Lambda_{(\bar{i}-1)}(\tau)$ (the last equality is our induction hypothesis). So, $v\in\Lambda_{(\bar{i}-1)}(\tau)$. Take a curve of vectors $v(t)$ in W such that $v(t)\in\Lambda_{(\bar{i}-1)}(t)$ for any t close to τ and $v(\tau)=v$. Then by induction hypothesis $\omega(v(t),w(t))=0$ for any t close to τ. Differentiating the last identity at $t=\tau$ and using (3.6) we get that $\omega\big(v'(\tau),w\big)=0$. Since the last identity holds for any $w\in\Lambda^{(\bar{i}-1)}(\tau)$, then $v'(\tau)\in\Big(\Lambda^{(\bar{i}-1)}(\tau)\Big)^{\angle}=\Lambda_{(\bar{i}-1)}(\tau)$ (the last equality is our induction hypothesis). So, $v\in\Lambda_{(\bar{i})}(\tau)$, which implies the inclusion $\Big(\Lambda^{(\bar{i})}(\tau)\Big)^{\angle}\subset\Lambda_{(\bar{i})}(\tau)$. The proof of the lemma is completed. $\square$

3.2. **Filling the Young diagram D by bases of $\Lambda(t)$.** As before, assume that the reduced diagram Δ of the curve consists of d level, the number of superboxes in the ith level of the diagram Δ is equal to p_i, and their sizes are equal to r_i. By our assumptions $\Lambda^{(p_1)}(t)=W$, which together with (3.4) implies that

$$\Lambda_{(p_1)}(t)=0,\quad \dim\Lambda_{(p_1-1)}(t)=r_1. \tag{3.7}$$

Denote also by σ_i the special (i.e. the last) superbox of the ith level of Δ. From the second relation of (3.3) it follows that

$$\Big(\Lambda_{(p_i)}\Big)^{(1)}(t)\subseteq\Lambda_{(p_i-1)}(t),\quad \forall\, 1\leq i\leq q \tag{3.8}$$

For any $1\leq i\leq d$ choose a complement $\widetilde{V}_{\sigma_i}(t)$ of the subspace $\Big(\Lambda_{(p_i)}\Big)^{(1)}(t)$ in the space $\Lambda_{(p_i-1)}(t)$ (smoothly w.r.t. t):

$$\Lambda_{(p_i-1)}=\Big(\Lambda_{(p_i)}\Big)^{(1)}(t)\oplus\widetilde{V}_{\sigma_i}(t). \tag{3.9}$$

Note that from (3.7) it follows that $\widetilde{V}_{\sigma_1}(t)=\Lambda_{(p_1-1)}(t)$. Let $\widetilde{\Delta}$ be the diagram, obtained from Δ by joining to Δ one more column from the left, having the same length as the first column of Δ. The boxes of $\widetilde{\Delta}$ will be called superboxes as well. For any $1\leq i\leq d$ take a tuple of vectors

$E_{\sigma_i}(t)$, constituting a basis of $\widetilde{V}_{\sigma_i}(t)$ (smoothly in t). Then to any superbox of $\tilde{\Delta}$ we will assign a tuple of vectors in the following way

$$E_{l^j(\sigma_i)}(t) \stackrel{def}{=} E_{\sigma_i}^{(j)}(t), \quad \forall\, 0 \le j \le p_i, \tag{3.10}$$

where l is the left shift on the diagram $\widetilde{\Delta}$.

Lemma 2. *Assume that a superbox $a \in \widetilde{\Delta}$ lies in the $j(a)$th column and $i(a)$th level of the diagram $\widetilde{\Delta}$ and let Ov_a be the set of all superboxes, lying over a in the column of a. Then the following relations hold*

$$\begin{gathered} \{E_a(t)\} \bigcap \Big(\Big(\bigoplus_{b \in \mathrm{Ov}_a} \mathrm{span}\{E_b(t)\}\Big) \oplus \Lambda_{(j(a)-1)}(t)\Big) = 0, \\ \dim \mathrm{span}\{E_a(t)\} = \dim \mathrm{span}\{E_{\sigma_{i(a)}}(t)\} = r_{i(a)}. \end{gathered} \tag{3.11}$$

Proof. Let $\prec$ be the order on the set of superboxes of the diagram $\widetilde{\Delta}$, defined as follows: $b_1 \prec b_2$ if either b_1 is higher than b_2 in $\widetilde{\Delta}$ or they are on the same level, but b_1 is located from the right to b_2 (or, equivalently, either $i(b_1) < i(b_2)$ or $i(b_1) = i(b_2)$, but $j(b_1) > j(b_2)$). Let us prove (3.11) by induction on the set of superboxes of the diagram $\widetilde{\Delta}$ with the introduced order $\prec$. For $a = \sigma_1$ relations (3.11) follow immediately from (3.7). Now assume that (3.11) is true for any superbox $a \in \widetilde{D}$ such that $a \prec \sigma$ and prove it for $a = \sigma$. We have the following two cases:

1. The superbox σ is special. In this case by induction hypothesis it is easy to show that

$$\Big(\bigoplus_{b \in \mathrm{Ov}_\sigma} \mathrm{span}\{E_b(t)\}\Big) \oplus \Lambda_{(p_{i(\sigma)})}(t) = \Big(\Lambda_{(p_{i(\sigma)})}\Big)^{(1)}(t) \tag{3.12}$$

This together with (3.9) and the definitions of the numbers r_i implies (3.11) for $a = \sigma$.

2. The superbox σ is not special. Using our induction assumptions we can choose a subspace $C(t)$ of $\Lambda_{(j(\sigma)-1)}(t)$ smoothly w.r.t. t such that

$$\Lambda_{(j(\sigma)-1)}(t) = \Big(\bigoplus_{b \in \mathrm{Ov}_{r(\sigma)}} \mathrm{span}\{E_b(t)\}\Big) \oplus \mathrm{span}\{E_{r(\sigma)}(t)\} \oplus \Lambda_{(j(\sigma))}(t) \oplus C(t), \tag{3.13}$$

where as before $r(\sigma)$ is the superbox, located from the right to σ in $\widetilde{\Delta}$.

From (3.2), the first relation of (3.3), and (3.13) it follows that

$$\begin{gathered} \dim(\Lambda_{(j(\sigma)-1)})^{(1)}(t) - \dim \Lambda_{(j(\sigma)-1)}(t) = \dim \Lambda_{(j(\sigma)-1)}(t) - \dim \Lambda_{(j(\sigma))}(t) = \\ \sum_{b \in \mathrm{Ov}_{r(\sigma)} \cup r(\sigma)} \dim \mathrm{span}\{E_b(t)\} + \dim C(t) = \sum_{k=1}^{i(\sigma)} r_k + \dim C(t). \end{gathered} \tag{3.14}$$

On the other hand, using definitions (2.1), (3.1), (3.10), (3.13), the induction hypothesis, and relation (3.2) one gets easily that

$$\begin{gathered} \dim(\Lambda_{(j(\sigma)-1)})^{(1)}(t) - \dim \Lambda_{(j(\sigma)-1)}(t) \le \sum_{k=1}^{i(\sigma)-1} r_k + \big(\dim \mathrm{span}\{E_{r(\sigma)}(t), E_\sigma(t)\} - \\ \dim \mathrm{span}\{E_{r(\sigma)}(t)\}\big) + \big(\dim C^{(1)}(t) - \dim C(t)\big) \le \sum_{k=1}^{i(\sigma)} r_k + \dim C(t). \end{gathered} \tag{3.15}$$

If for $a = \sigma$ one of the identities in (3.11) does not hold, then in the chain of the inequalities (3.15) there is at least one strong inequality, which is in the contradiction with (3.14). So, the identities (3.11) are valid for $a = \sigma$, which completes the proof of (3.11) by induction. □

Let $\mathcal{F}_k$ be the kth column of the diagram Δ. From Lemma 2 it follows easily the following

Corollary 1. *The following splittings hold for any* $0 \le j \le p_1$

$$\Lambda_{(j)}(t) = \bigoplus_{a \in \bigcup_{s=j+1}^{p_1} \mathcal{F}_s} \operatorname{span}\{E_a(t)\}, \quad \left(\Lambda_{(j)}\right)^{(1)}(t) = \bigoplus_{a \in \bigcup_{s=j+1}^{p_1} \mathcal{F}_s \cup l(F_{j+1})} \operatorname{span}\{E_a(t)\} \tag{3.16}$$

In particular, $\Lambda(t) = \bigoplus_{a\in\Delta} \operatorname{span}\{E_a(t)\}$.

One can imagine that we fill the diagram Δ (or the original diagram D) by columns $E_a(t)^T$ by choosing bases of the subspaces $\widetilde{V}_{\sigma_i}$, satisfying (3.9), and by differentiating these bases as in (3.10). Tuples $\{E_a(t)\}_{a\in\Delta}$, obtained in this way, will be called *fillings of the Young diagram D, associated with the curve* $\Lambda(\cdot)$. The flag $0 = \Lambda_{(p_1}(t) \subset \Lambda_{(p_1-1}(t) \ldots \subset \Lambda_{(0)}(t) = \Lambda(t)$ can be recovered from this filling by the first relation of (3.16). In particular, this flag (and therefore the curve $\Lambda(\cdot)$ itself) can be recovered from the curves $t \mapsto V_{\sigma_i}(t)$, $1 \le i \le d$ by taking the corresponding extensions of them.

3.3. **The canonical complement of $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ and the canonical Euclidean structure on it.** In the previous subsection we took some complements $\widetilde{V}_{\sigma_i}$ of the subspaces $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in the spaces $\Lambda_{(p_i-1)}(t)$. In the present section we will show that such complements can be chosen canonically for a curve $\Lambda(t)$ with the Young diagram D, satisfying the following additional assumption:

Condition (G) *For any $1 \le i \le d-1$ and any t the rank of the restriction of the quadratic form $\dot{\Lambda}(t)$ to the subspace $\left(\Lambda_{(p_i-1)}\right)^{(p_i-1)}(t)$ is equal to* $\displaystyle\sum_{k=1}^{i} r_k$,

$$\forall 1 \le i \le d-1 \text{ and } \forall\, t: \quad \operatorname{rank}\left(\dot{\Lambda}(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) = \sum_{k=1}^{i} r_k. \tag{3.17}$$

Since $\operatorname{Ker}\dot{\Lambda}(t) = \Lambda_{(1)}(t)$ and $\Lambda_{(p_i-1)})^{(p_i-2)}(t) \subset \Lambda_{(1)}(t)$ (as a consequence of (3.3)), any curve $\Lambda(t)$ with the Young diagram D satisfies: $\operatorname{rank}\left(\dot{\Lambda}(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) \le \sum_{k=1}^{i} r_k$ for any $1 \le i \le d$. It implies easily that germs of curves, satisfying condition (G), are generic among all germs of curves with given Young diagram D. Besides, it is clear that curves with rectangular Young diagram satisfy condition (G) automatically (condition (G) is empty in this case).

Lemma 3. *Any monotonic curve $\Lambda(t)$ with the Young diagram D satisfies condition (G).*

Proof. For definiteness, assume that the curve $\Lambda(t)$ is monotonically nondecreasing. Take a filling $\{E_a(t)\}_{a\in\Delta}$ of the Young diagram D, associated with the curve $\Lambda(\cdot)$. Let

$$Z_i(t) = \operatorname{span}\{E_{\sigma_k}^{(p_k-1)}(t)\}_{k=1}^{i}, \quad 1 \le i \le q. \tag{3.18}$$

It is clear that $\{Z_i(t)\}_{i=1}^{d}$ is a monotonically increasing (by inclusion) sequence of subspaces for any t. As a consequence of Lemma 2, we have

$$\dim Z_i(t) = \sum_{k=1}^{i} r_k, \tag{3.19}$$

$$(\Lambda_{(p_i-1)})^{(p_i-1)}(t) = \left((\Lambda_{(p_i-1)})^{(p_i-1)}(t) \cap \Lambda_{(1)}(t)\right) \oplus Z_i(t) \tag{3.20}$$

Since $\operatorname{Ker}\dot\Lambda(t) = \Lambda_{(1)}(t)$, we get from (3.20) that

$$\operatorname{rank}\left(\dot\Lambda(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}\right) = \operatorname{rank}\left(\dot\Lambda(t)|_{Z_i(t)}\right). \tag{3.21}$$

Besides, from monotonicity the quadratic form $\dot\Lambda(t)|_{Z_d(t)}$ is positive definite. Hence, the quadratic forms $\dot\Lambda(t)|_{Z_i(t)}$ are positive definite as well. Then the lemma follows form relations (3.19) and (3.21). $\square$

Now define the following subspaces of the ambient symplectic space W:

$$W_i(t) = (\Lambda_{(p_1-1)}(t))^{(2p_1-1)} + (\Lambda_{(p_2-1)}(t))^{(2p_2-1)} + \ldots + (\Lambda_{(p_i-1)}(t))^{(2p_i-1)}. \tag{3.22}$$

Lemma 4. *If a curve $\Lambda(t)$ with the Young diagram D satisfies condition (G), then for any $1 \le i \le d$ the restriction of the symplectic form ω to the subspace $W_i(t)$ is nondegenerated and $dim W_i = 2\sum_{k=1}^{i} p_k r_k$.*

Proof. The proof of the lemma is by induction w.r.t. i. First let us introduce some notations. Let $\overline{\Delta}$ be the diagram obtained from Δ by the reflection w.r.t. its left edge. We will work with the diagram $\Delta\cup\overline{\Delta}$, which is symmetric w.r.t. the left edge of the diagram Δ. Similar to above, we will denote by l the left shift on the diagram $\Delta\cup\overline{\Delta}$. If S is a subset of the diagram Δ, we will denote by $\bar S$ the subset of $\bar\Delta$, obtained by the reflection of S w.r.t. the left edge of Δ. Also in the sequel, given two tuples of vectors $V_1 = (v_{11}, \ldots, v_{1n_1})$ and $V_2 = (v_{21}, \ldots, v_{2n_2})$ by $\omega(V_1, V_2)$ we will mean the $n_1 \times n_2$-matrix with the (i,j)-entry equal to $\omega(v_{1i}, v_{2j})$. Take a filling $\{E_a(t)\}_{a\in\Delta}$ of the Young diagram D, associated with the curve $\Lambda(\cdot)$. Define tuples E_a also for $a \in \bar\Delta$ in the following way: $E_{l^j(a_i)} = E_{a_i}^{(j)}(t)$, $1 \le j \le p_i$, where, as before, a_i is the first superbox in the ith level Υ_i of Δ. By definition $W_i(t) = \operatorname{span}\{E_a(t)\}_{a\in\cup_{k=1}^i \Upsilon_k\cup\bar\Upsilon_k}$.

1. Let us prove the lemma for $i = 1$. By condition (G) the matrix $\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-1)}(t)\big)$ is nonsingular. On the other hand, since $\Lambda_{(1)}(t) = \left(\Lambda^{(1)}(t)\right)^{\angle}$, one has $\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-2)}(t)\big) \equiv 0$. Differentiating the last identity, we get $\omega\big(E_{\sigma_1}^{(p_1+1)}(t), E_{\sigma_1}^{(p_1-2)}(t)\big) = -\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-1)}(t)\big)$. In the same way, using (3.4), it is easy to obtain that

$$\omega\big(E_{\sigma_1}^{(p_1+i)}(t), E_{\sigma_1}^{(p_1-i-1)}(t)\big) = (-1)^i\omega\big(E_{\sigma_1}^{(p_1)}(t), E_{\sigma_1}^{(p_1-1)}(t)\big).$$

In particular, all matrices $\omega\big(E_{\sigma_1}^{(p_1+i)}(t), E_{\sigma_1}^{(p_1-i-1)}(t)\big)$ are nonsingular. Therefore the matrix with the entries, which are equal to the value of the form ω on all pairs of vectors from the tuple $\{E_a(t)\}_{a\in\Upsilon_1\cup\bar\Upsilon_1}$, is block-triangular w.r.t. the nonprincipal diagonal with nonsingular blocks on the nonprincipal diagonal. This implies that the tuple $\{E_a(t)\}_{a\in\Upsilon_1\cup\bar\Upsilon_1}$ constitutes the basis of W_1 and the form $\omega|_{W_1}$ is nondegenerated, which completes the proof of the statement of the lemma in the case $i = 1$.

2. Now assume that the statement of the lemma holds for $i = i_0 - 1$ and prove it for $i = i_0$. Let Δ_i be the subdiagram of Δ, consisting of the first i rows of Δ, $\Delta_i = \bigcup_{k=1}^{i}\Upsilon_k$. Divide the diagram $\Delta_{i_0}\cup\bar\Delta_{i_0}$ on four parts $\{A_k\}_{k=1}^4$: A_1 is a union of the last $p_1 - p_{i_0}$ columns of the diagram Δ_{i_0}, A_2 is obtained by the reflection of A_1 w.r.t. the left edge of Δ_{i_0}, i.e. $A_2 = \bar A_1$, $A_3 = \Delta_{i_0-1}\backslash(A_1\cup A_2)$, and $A_4 = \Upsilon_{i_0}$.

Set $C_k(t) = \operatorname{span}\{E_a(t)\}_{a\in A_k}$, $k = 1, \ldots, 4$. Note that from (3.16) it follows that $C_1(t) = \Lambda_{(p_{i_0})}(t)$. By constructions $W_{i_0}(t) = C_1(t) + C_2(t) + C_3(t) + C_4(t)$ and $W_{i_0-1} = C_1(t) + C_2(t) +$

$C_3(t)$. Moreover, by induction hypothesis

$$W_{i_0-1}(t) = C_1(t) \oplus C_2(t) \oplus C_3(t) \tag{3.23}$$

$$C_1(t)^{\angle} \cap W_{i_0-1}(t) = C_1(t) \oplus C_3(t). \tag{3.24}$$

The last two identities follow just from comparison of dimensions. Besides, using (3.4), one has also that

$$C_1 + C_3 + C_4 \subset C_1(t)^{\angle}. \tag{3.25}$$

Assume that $x \in \operatorname{Ker} \omega|_{W_{i_0}(t)}$, $x = \sum_{k=1}^{4} x_k$, where $x_k \in C_k(t)$. Then (3.25) implies that $\omega(v, x) = \omega(v, x_2) = 0$ for any $v \in C_1(t)$. This together with (3.23) and (3.24) yields that $x_2 = 0$.

Further, by the same arguments as in the proof of the case $i = 1$, applied for the tuple $\{E_a\}_{a \in \mathcal{F}_{p_{i_0}} \cap \Delta_{i_0}}$ instead of the tuple E_{σ_1}, one obtains from (3.17) for $i = i_0$ that $\omega|_{C_3(t)+C_4(t)}$ is nondegenerated and $\dim(C_3(t)+C_4(t)) = 2p_{i_0} \sum_{k=1}^{i_0} r_k$. The latter implies that $C_1(t) \cap C_3(t) = 0$. Besides, from (3.25) it follows that $\omega(v, x) = \omega(v, x_3 + x_4) = 0$ for any $v \in C_3(t) + C_4(t)$, which together with two previous sentences implies that $x_3 = x_4 = 0$. Therefore $x \in C_1(t) \subset W_{i_0-1}(t)$, which implies that $x = x_1 = 0$ by induction hypothesis. This yields that the form $\omega|_{W_{i_0}(t)}$ is nondegenerated. Moreover, from the same arguments it follows that the condition $\sum_{k=1}^{4} x_k = 0$ implies that $x_k = 0$ for any $1 \leq k \leq 4$. Hence $W_{i_0}(t) = C_1(t) \oplus C_2(t) \oplus C_3(t) \oplus C_4(t)$ and the statement of the lemma about the dimension of $W_{i_0}(t)$ holds. The proof of the lemma is completed. $\square$

Finally, let

$$V_i(t) = \Lambda_{(p_i-1)}(t) \cap W_{i-1}(t)^{\angle} \tag{3.26}$$

As a direct consequence of Lemma 4, we get that the subspace $V_i(t)$ is complementary to $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$,

$$\Lambda_{(p_i-1)}(t) = \left(\Lambda_{(p_i)}\right)^{(1)}(t) \oplus V_i(t). \tag{3.27}$$

The subspaces $V_i(t)$, defined by (3.26) will be called the *canonical complement of* $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ *in* $\Lambda_{(p_i-1)}(t)$. The following equivalent description of the subspaces $V_i(t)$ will be very useful in the sequel:

Lemma 5. *A sequence of subspaces* $\{\widetilde{V}_{\sigma_i}(t)\}_{i=1}^{d}$*, satisfying* (3.9)*, consists of the canonical complements of* $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ *in* $\Lambda_{(p_i-1)}(t)$ *for any* $1 \leq i \leq d$ *if and only if smooth (w.r.t. t) tuples of vectors* $E_{\sigma_i}(t)$*, constituting bases of* $\widetilde{V}_{\sigma_i}(t)$*, satisfy:*

$$\forall\, 1 \leq j < i \leq d \text{ and } \forall\, 1 \leq k \leq p_j - p_i + 1: \quad \omega\big(E_{\sigma_i}^{(p_i-1)}(t), E_{\sigma_j}^{(p_j-1+k)}(t)\big) = 0 \tag{3.28}$$

or, equivalently, taking into account notations in (3.10),

$$\forall\, 1 \leq j < i \leq d \text{ and } \forall\, 1 \leq k \leq p_j - p_i + 1: \quad \omega\big(E_{a_i}(t), E_{a_j}^{(k)}(t)\big) = 0. \tag{3.29}$$

The lemma can be easily proved by rewriting identity (3.26) in terms of bases $E_{\sigma_i}(t)$ and appropriate differentiations.

Further, it turns out that on each canonical complement $V_i(t)$ one can define the canonical quadratic form. Indeed, given a vector $v \in V_i(t)$ take a smooth curve $\varepsilon(t)$ in W such that

(1) $\varepsilon(\tau) = v$;

(2) $\varepsilon(t) \in V_i(t)$ for any t close to τ.

Then by our constructions it is easy to see that for any $0 \le j \le p_i - 1$

$$\varepsilon^{(j)}(\tau) \in \Lambda_{(p_i-1-j)}(\tau), \tag{3.30a}$$

$$\varepsilon^{(j+1)}(\tau) \notin \Lambda_{(p_i-1-j)}(\tau), \quad \text{if } v \ne 0, \tag{3.30b}$$

$$\varepsilon^{(j+1)}(\tau) \in \Lambda_{(p_i-1-j)}(\tau), \quad \text{if } v = 0 \tag{3.30c}$$

For this take a basis $E_{\sigma_i(t)}$ of $V_i(t)$, depending smoothly on t, expand our curve $\varepsilon(t)$ w.r.t. this basis, and use the fact that for any $0 \le j \le p_i - 1$

$$\bigoplus_{s=0}^{j} \operatorname{span}\{E_{\sigma_i}^{(s)}(t)\} \subset \Lambda_{(p_i-1-j)}(\tau), \quad \operatorname{span}\{E_{\sigma_i}^{(j+1)}(t)\} \cap \Lambda_{(p_i-1-j)}(\tau) = 0, \tag{3.31}$$

which is a direct consequence of Lemma 2. From (3.30a), (3.30c), the fact that $\Lambda(t)$ is the curve of Lagrangian subspaces, and the identity (3.4) it follows that

$$Q_{i,\tau}(v) = \omega\big(\varepsilon^{(p_i)}(\tau), \varepsilon^{(p_i-1)}(\tau)\big) \tag{3.32}$$

is a well defined quadratic form on $V_i(\tau)$, which does not depend on the choice of the curve $\varepsilon(\tau)$ satisfying conditions (1) and (2) above. The form $Q_{i,\tau}(v)$ will be called the *canonical quadratic form on* $V_i(\tau)$. The quadratic forms $Q_{i,\tau}(v)$ are nondegenerated for any $1 \le i \le d$. Indeed, if tuples $E_{\sigma_i(t)}$ constitute bases of $V_i(t)$ for any $1 \le i \le d$ and $Z_d(t)$ is as in (3.18), then from Lemma 5 it follows that the matrix of the quadratic form $\dot\Lambda(\tau)|_{Z_d(\tau)}$ in the basis $\{E_{\sigma_k}^{(p_k-1)}(\tau)\}_{k=1}^d$ is block-diagonal and the diagonal blocks are exactly the matrices of the forms $Q_{i,\tau}(v)$ in the bases $E_{\sigma_i(t)}$. Then the nondegenericity of the form $Q_{i,\tau}(v)$ follows from condition (G) and (3.21). Moreover, if the curve $\Lambda(t)$ is monotonically nondecreasing, then the forms $Q_{i,\tau}$ are positive definite. In this case the Euclidean structure on $V_{\sigma_i}(\tau)$, corresponding to the form $Q_{i,\tau}$ will be called the *canonical Euclidean structure on* $V_i(\tau)$.

From now on for simplicity of presentation we will assume that the curve $\Lambda(t)$ is monotonically nondecreasing. All necessary changes in the formulation of the results for nonmonotonic curves, satisfying condition (G), will be indicated in section 4. For any $1 \le i \le d$, let $\mathfrak{B}_i$ be a fiber bundle over the curve $\Lambda(t)$ such that the fiber of $\mathfrak{B}_i$ over the point $\Lambda(t)$ consists of all orthonormal bases of the space $V_{\sigma_i}(t)$ w.r.t. the canonical Euclidean structure on $V_i(t)$. Note that $\mathfrak{B}_i$ is the principle bundle with the structure group $O(r_i)$.

3.4. **The canonical connections on the bundles $\mathfrak{B}_i$.**

Now let us prove the following

Proposition 1. *Each bundle $\mathfrak{B}_i$ is endowed with the* canonical principal connection *uniquely characterized by the following condition: the section $E_{\sigma_i}(t)$ of $\mathfrak{B}_i$ is horizontal w.r.t. this connection if and only if* $\operatorname{span}\{E_{\sigma_i}^{(p_i)}(t)\}$ *are isotropic subspaces of W for any t. Given any two horizontal sections $E_{\sigma_i}(t)$ and $\widetilde{E}_{\sigma_i}(t)$ of $\mathfrak{B}_i$ there exists a constant orthogonal matrix U_i such that*

$$\widetilde{E}_{\sigma_i}(t) = E_{\sigma_i}(t) U_i. \tag{3.33}$$

Proof. As in the proof of Lemma 4, given two tuples of vectors $V_1 = (v_{11}, \dots, v_{1n_1})$ and $V_2 = (v_{21}, \dots, v_{2n_2})$ by $\omega(V_1, V_2)$ we will mean the $n_1 \times n_2$-matrix with the (i,j)-entry equal to $\omega(v_{1i}, v_{2j})$. With this notation, it is obvious that if $V_i = \operatorname{span}\{\widetilde{E}_{\sigma_i}\}$, then the subspace $\operatorname{span}\{\widetilde{E}_{\sigma_i}^{(p_i)}(t)\}$ is isotropic if and only if

$$\omega\big(\widetilde{E}_{\sigma_i}^{(p_i)}(t), \widetilde{E}_{\sigma_i}^{(p_i)}(t)\big) = 0. \tag{3.34}$$

Note also that from definition of the canonical Euclidean structure it follows immediately that for any section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ the following identity holds

$$\omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i-1)}(t)\big) = \mathrm{Id}. \tag{3.35}$$

Take any two section $E_{\sigma_i}(t)$ and $\widetilde{E}_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$. Then there exists a curve $U_i(t)$ of orthonormal matrices such that $\widetilde{E}_{\sigma_i}(t) = E_{\sigma_i}(t)U_i(t)$. Using relation $\Lambda_{(1)}(t) = \big(\Lambda^{(1)}(t)\big)^\angle$ and formula (3.35), it is easy to get that

$$\omega\big(\widetilde{E}_{\sigma_i}^{(p_i)}(t), \widetilde{E}_{\sigma_i}^{(p_i)}(t)\big) = U(t)^T\left(2p_iU'(t) + \omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)U(t)\right).$$

So, relation (3.34) holds if and only the matrix $U(t)$ satisfies the following differential equation

$$2p_iU'(t) + \omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)U(t) = 0. \tag{3.36}$$

Note that the matrix $\omega\big(E_{\sigma_i}^{(p_i)}(t), E_{\sigma_i}^{(p_i)}(t)\big)$ is antisymmetric. So, equation (3.36) has solutions in $O(r_i)$, which are defined up to the right translation there. This completes the proof of the proposition. $\square$

Now, if for any $1 \le i \le d$ we take a horizontal section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ and set , as before, $E_{l^j(\sigma_i)}(t) = E_{\sigma_i}^{(j)}(t)$ for $0 \le j \le p_i - 1$, then from (3.33) it follows that for any superbox a the subspaces $V_a(t) = \mathrm{span}\{E_a(t)\}$ do not depend on the choice of a horizontal sections $E_{\sigma_i}(t)$. Moreover, from this and Lemma 2 we get the *canonical splitting* $\Lambda(t) = \bigoplus_{a\in\Delta} V_a(t)$ *of the subspaces* $\Lambda(t)$.

3.5. **The completion of horizontal sections to quasi-normal moving frames.** In the sequel it will be more convenient to use the following obviously equivalent description of quasi-normal mappings:

Lemma 6. *A symmetric compatible mapping* $R : \Delta \times \Delta \mapsto \mathrm{Mat}$ *is quasi-normal if and only if the following four conditions hold:*

1. *If a and b are two consecutive superboxes in the same level of Δ, then the matrix $R(a, b)$ is antisymmetric;*
2. *If both superboxes a and b are not special and do not lie in the same or adjacent columns, then $R(a, b) = 0$;*
3. *If both superboxes a and b are not special, lie in the adjacent (but not the same) columns and one of the superboxes is located from below and from the left w.r.t. the other, then $R(a, b) = 0$;*
4. *If a superbox a is special, a superbox b is not special and b is located from the left to a, but not in the adjacent column, then $R(a, b) = 0$.*

Further, for all $1 \le i \le d$, fix a horizontal section $E_{\sigma_i}(t)$ of the bundle $\mathfrak{B}_i$ and complete it to the moving basis $\{E_a(t)\}_{a\in\Delta}$ of $\Lambda(t)$ setting, as before, $E_{l^j(\sigma_i)}(t) = E_{\sigma_i}^{(j)}(t)$ for $0 \le j \le p_i - 1$. Also let

$$F_{a_i}(t) = E_{a_i}'(t). \tag{3.37}$$

From the definition of the canonical Euclidean structure it follows that $\omega(F_{a_i}(t), E_{a_i}(t)) = Id$. From the normalization conditions (3.29) with $k = 1$ it follows that $\omega(F_{a_i}(t), E_{a_j}(t)) = 0$ for any $i \ne j$. Further, by definition of the horizontal section of the bundle $\mathfrak{B}_i$ one has $\omega(F_{a_i}(t), F_{a_i}(t)) = 0$. Finally, from the normalization conditions (3.29) with $k = 2$ it follows that $\omega(F_{a_i}(t), F_{a_j}(t)) = 0$ for $i \ne j$ as well. Combining all these identities with the fact that the subspaces $\Lambda(t)$ are Lagrangian and the relation $\Lambda_{(1)}(t) = \big(\Lambda^{(1)}(t)\big)^\angle$, we get that the tuple

$\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$, where, as before, $\mathcal{F}_1$ denotes the first column of Δ, does not contradict the relations for a Darboux frame. Besides, by our constructions it satisfies first two equations of (2.7). In this subsection we prove the following

Proposition 2. *The tuple* $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$ *can be uniquely completed to a quasi-normal moving frame of the curve* $\Lambda(t)$.

Proof. Take a tuple $\{F_b(t)\}_{b\in\Delta\setminus\mathcal{F}_1}$, which completes the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$ to a moving Darboux's frame in W. Then from the definition of Darboux's frame and the first two equations of (2.7) it follows that this moving Darboux frame have the structural equation (2.7) for some symmetric mappings $R_t : \Delta\times\Delta \mapsto \mathrm{Mat}$ compatible with the Young diagram D. As before, denote by $\mathcal{F}_j$ the jth column of Δ, $1 \le j \le p_1$. Our proposition will follow from the following

Statement 1. *For any* $1 \le k \le p_1$ *there exists a unique tuple of columns of vectors*

$$\{F_b(t) : b \in \bigcup_{j=1}^{k} \mathcal{F}_j\}$$

such that the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t) : b \in \bigcup_{j=1}^{k} \mathcal{F}_j\}\big)$ *can be completed to a moving Darboux frame* $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\Delta}\big)$ *such that if the mapping* $R_t : \Delta\times\Delta \mapsto \mathrm{Mat}$ *appears in the structural equation* (2.7) *for this moving frame, then the mapping* R_t *satisfies conditions (1)-(4) of Lemma 6 for any pair* (a, b) *with at least one superbox belonging to the first* $(k-1)$ *columns of* Δ.

Indeed, our proposition is just Statement 1 in the case $k = p_1$ (the only pair of superboxes, which is not covered by Statement 1, is (σ_1, σ_1), where, as before, σ_1 is the special (the last) superbox of the first level, but this pair does not satisfy any of conditions (1)-(4) of Lemma 6).

We will prove Statement 1 by induction w.r.t. k. For $k = 1$ there is nothing to prove, because the tuple $\{F_c\}_{c\in\mathcal{F}_1}$ is uniquely determined by the second line of (2.7) (which together with the first line of (2.7) is equivalent to (3.37)), while the Statement 1 for $k = 1$ does not impose any conditions on the symmetric compatible mapping R_t, appearing in (2.7).

Now suppose that Statement 1 is proved for some $k = \bar{k}$, where $1 \le \bar{k} \le p_1 - 1$, and prove it for $k = \bar{k} + 1$. Let $\{F_b(t) : b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$ be the tuple, satisfying Statement 1 for $k = \bar{k}$. Take a tuple $\{F_b(t) : b \in \Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$, which completes the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t) : b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}\big)$ to a moving Darboux's frame in W and assume that $R_t : \Delta\times\Delta \mapsto \mathrm{Mat}$ is the mapping, appearing in the structural equation for this frame. If $\{\widehat{F}_b(t) : b \in \Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}$ is another tuple, completing the tuple $\big(\{E_a\}_{a\in\Delta}, \{F_b(t)\}_{b\in\mathcal{F}_1}\big)$ to a moving Darboux's frame in W, then there exists a symmetric mapping $\Gamma_t : (\Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \times (\Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \mapsto \mathrm{Mat}$, compatible with the diagram, obtained from D by erasing the first $\bar{k}$ column, such that

$$\forall a \in \Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \quad \widehat{F}_a(t) = F_a(t) + \sum_{b\in\Delta\setminus\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j} E_b(t)\Gamma_t(a, b). \tag{3.38}$$

Suppose that $\widehat{R}_t : \Delta\times\Delta \mapsto \mathrm{Mat}$ is the symmetric mapping compatible with the Young diagram D such that similarly to last two equations of (2.7) one has

$$
(3.39)\qquad \begin{cases} F_a'(t) = \sum_{b\in\Delta} \widehat{E}_b R_t(a,b) - \widehat{F}_{r(a)} & \text{if } a \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \\ \widehat{F}_a'(t) = \sum_{b\in\Delta} \widehat{E}_b R_t(a,b) - \widehat{F}_{r(a)} & \text{if } \ a \in \Delta\backslash(\bigcup_{j=1}^{\bar{k}} \mathcal{F}_j \cup \mathcal{S}) \\ \widehat{F}_a'(t) = \sum_{b\in\Delta} \widehat{E}_b R_t(a,b) & \text{if } \ a \in \mathcal{S}, \end{cases}
$$

(note that from the first line of (3.39), one has $\widehat{R}_t(a,b) = R_t(a,b)$, if at least one of the superboxes (a,b) belongs to the first $\bar{k}$ columns of Δ). Let us extend the mappings $\Gamma_t : (\Delta\backslash \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \times (\Delta\backslash \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j) \mapsto \mathrm{Mat}$ to the symmetric mapping, denoted by the same letter Γ_t, from $\Delta\times\Delta$ to Mat compatible with the diagram D, by setting

$$
(3.40)\qquad \Gamma_t(a,b)(t) = \Gamma_t(b,a)^T = 0, \quad \forall\, b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j, a\in\Delta.
$$

Then, substituting (3.38) into two last lines of (3.39) and using (2.7), one can easily obtain

$$
(3.41)\qquad \widehat{R}_t(a,b) = R_t(a,b) + \frac{d}{dt}\Gamma_t(a,b) + \Gamma_t\big(a,r(b)\big) + \Gamma_t\big(r(a),b\big),
$$

where the term $\Gamma_t(a,r(b))$ is omitted, if b is special, and the term $\Gamma_t(r(a),b)$ is omitted, if a is special. Using transformation rule (3.41), we will prove the following

Statement 2. *There exists the unique choice of matrices $\Gamma_t\big(\tilde{a},\tilde{b}\big)$ with at least one of the superboxes belonging to the $(\bar{k}+1)$th column of Δ and the other one lying from the right to the $\bar{k}$th column of Δ such that the matrix $\widehat{R}_t(a,b)$ satisfies all conditions (1)-(4) of Lemma 6 for any pairs (a,b) with at least one of the superboxes belonging to the $\bar{k}$th column of Δ and the other one lies from the right to the $(\bar{k}-1)$th column of Δ*

It is clear that Statement 2, relation (3.38), and the induction hypothesis will imply Statement 1 for $k = \bar{k}+1$. Let us prove statement 2. Suppose that $a\in\mathcal{F}_{\bar{k}}$. Then from (3.40) it follows that $\frac{d}{dt}\Gamma_t(a,b) = 0$ and $\Gamma_t\big(a,r(b)\big) = 0$. So, relations (3.41) in this case have a form

$$
(3.42)\qquad \widehat{R}_t(a,b) = R_t(a,b) + \Gamma_t\big(r(a),b\big),
$$

where the term $\Gamma_t\big(r(a),b\big)$ is omitted, if a is special (obviously it happens, when the level of a consists of only one superbox). Therefore, according to (3.42), if a is special or $b \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$ we have $\widehat{R}_t(a,b) = R_t(a,b)$, i.e. the matrix $R_t(a,b)$ is already independent of the choice of the complement of $\big(\{E_{\tilde{a}}\}_{\tilde{a}\in\Delta}, \{F_{\tilde{b}}(t) : \tilde{b} \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}\big)$ to a moving Darboux frame.

Now assume that a is not special and $b \notin \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$. Then there are the following three cases:

a) $b \notin \bigcup_{j=1}^{\bar{k}+1} \mathcal{F}_j$, i.e. b is not in the first $\bar{k}+1$ columns of Δ. Then the matrix $\Gamma_t\big(r(a),b\big)$ appears only ones in all relations,

$$
(3.43)\qquad \widehat{R}_t(\tilde{a},\tilde{b}) = R_t(\tilde{a},\tilde{b}) + \Gamma_t\big(r(\tilde{a}),\tilde{b}\big),
$$

where $\tilde{a}$ runs over the whole $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ. Putting

$$\Gamma_t\big(r(a), b\big) = -R_t(a, b), \tag{3.44}$$

we get $\widehat{R}_t(a, b) = 0$ for any $a \in \mathcal{F}_{\bar{k}}$, which corresponds to conditions (2) and (4) of Lemma 6, if b is not from the left to a. Obviously, the choice of $\Gamma_t\big(r(a), b\big)$ as in (3.44) is the unique one with these properties.

b) $b \in \mathcal{F}_{\bar{k}+1}$, but $b \neq r(a)$, i.e. b lies in the $(\bar{k}+1)$th column of Δ, but it is not in the same row with a. Let $a_1 = l(b)$. Then from the symmetricity of the mapping Γ_t (i.e. the relation $\Gamma_t(a, b) = \big(\Gamma_t(a, b)\big)^T$) it follows that the matrix $\Gamma_t\big(r(a_1), r(a)\big)$ appears twice in all relations (3.43), where $\tilde{a}$ runs over the $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ and $\tilde{b}$ runs over the $(\bar{k}+1)$th column $\mathcal{F}_{\bar{k}+1}$ of Δ. Namely, substituting $(\tilde{a}, \tilde{b}) = \big(r(a), a_1\big)$ into (3.43) and using the symmetricity of the mapping Γ_t we will get the following relation in addition to (3.42) (with $b = r(a_1)$):

$$\widehat{R}_t\big(a_1, r(a)\big) = R_t\big(a_1, r(a)\big) + \Gamma_t\big(r(a), r(a_1)\big)^T. \tag{3.45}$$

Hence, from symmetricity again we have

$$\widehat{R}_t\big(a, r(a_1)\big) - \widehat{R}_t\big(r(a), a_1\big) = R_t\big(a, r(a_1)\big) - R_t\big(r(a), a_1\big),$$

i.e. the matrix $R_t\big(a, r(a_1)\big) - R_t\big(r(a), a_1\big)$ does not depend on the choice of the complement of $\big(\{E_{\tilde{a}}\}_{\tilde{a}\in\Delta}, \{F_{\tilde{b}}(t) : \tilde{b} \in \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j\}\big)$ to a moving Darboux frame. Besides, for any pair of superboxes (a, a_1), $a \neq a_1$ in the $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ by an appropriate choice of $\Gamma_t\big(r(a), r(a_1)\big)$ we cannot "kill" both matrices $R_t\big(r(a), a_1\big)$ and $R_t\big(a, r(a_1)\big)$, but only one of them. We choose the following normalization: $\widehat{R}(a, r(a_1)) = 0$, if a_1 is higher than a. We can do it by putting $\Gamma_t\big(r(a), r(a_1))\big) = -R_t\big(a, r(a_1)\big)$. This normalization corresponds to conditions (3) of Lemma 6. Obviously, such choice of $\Gamma_t\big(r(a), r(a_1)\big)$ is the unique one with these properties.

c) $b = r(a)$. Then the matrix $\Gamma_t\big(r(a), r(a)\big)$ appears only once in all relations (3.43) where $\tilde{a}$ runs over the whole $\bar{k}$th column $\mathcal{F}_{\bar{k}}$ of Δ, namely

$$\widehat{R}_t(a, r(a)) = R_t(a, r(a)) + \Gamma_t\big(r(a), r(a)\big). \tag{3.46}$$

On the other hand, by our assumptions $\Gamma_t\big(r(a), r(a)\big)$ should be symmetric. Therefore, using (3.46), we cannot "kill" the whole matrix $R_t(a, r(a))$, but only its symmetric part (by putting $\Gamma_t\big(r(a), r(a)\big) = -\frac{1}{2}\big(R_t(a, r(a)) + R_t(a, r(a))^T\big)\big)$. It corresponds to conditions (1) of Lemma 6 with $a \in \mathcal{F}_{\bar{k}}$. Obviously, such choice of $\Gamma_t\big(r(a), r(a)\big)$ is the unique one with these properties. In this way we have found uniquely all matrices $\Gamma_t(\tilde{a}, \tilde{b})$ with $\tilde{a} \in \mathcal{F}_{\bar{k}+1}$, $b \notin \bigcup_{j=1}^{\bar{k}} \mathcal{F}_j$ such that the matrix $\widehat{R}_t(a, b)$ satisfies all conditions (1)-(4) of Lemma 6 for any pairs (a, b), where $a \in \mathcal{F}_{\bar{k}}$, $b \notin \bigcup_{j=1}^{\bar{k}-1} \mathcal{F}_j$. Taking $\Gamma_t(\tilde{b}, \tilde{a}) = \Gamma_t(\tilde{a}, \tilde{b})^T$, we will have the same properties for $\widehat{R}_t(b, a)$ with a and b as in the previous sentence. This completes the proof of Statement 2, therefore also the proof of the Statement 1 for $k = \bar{k}+1$, and then by induction the proof of Proposition 2. $\square$

3.6. **Normality of the obtained quasi-normal moving frames.** In the present subsection we will show that the quasi-normal moving frame, constructed in the previous subsection, is in fact a normal moving frame. Note that in the previous subsection we did not use at all the normalization conditions (3.29) with $k \geq 3$. As before, we denote by d the number of levels in the diagram Δ, by p_i the number of superboxes in the ith level, and by a_i the first superbox in the ith level. The normality of the constructed quasinormal frame will obviously follow from the following

Proposition 3. *A quasi-normal moving frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ is normal if and only if conditions (3.29) hold for any $1 \le j < i \le d$ and $3 \le k \le p_j - p_i + 1$.*

Proposition 3 will follow by induction from the following

Statement 3. *Fix $s \in \mathbb{N}$ and let $R_t : \Delta \times \Delta \to \mathrm{Mat}$ be a quasi-normal mapping, satisfying the following condition: for any i and j, $1 \le j < i \le d$, the matrix $R_t(a,b) \equiv 0$ for all first $\min\{s-1, p_j - p_i - 1\}$ pairs (a,b) in the tuple (2.5). Then for any i and j, $1 \le j < i \le d$, such that $1 \le s \le p_j - p_i$, the sth pair $(\bar a_i^s, \bar a_j^s)$ of the tuple (2.5) satisfies*

$$R_t(\bar a_i^s, \bar a_j^s) = \pm\omega\big(E_{a_j}^{(s+2)}(t), E_{a_i}(t)\big). \tag{3.47}$$

Before proving Statement 3, let us introduce some notations. As in the proof of Lemma 4, let $\overline{\Delta}$ be the diagram obtained from Δ by the reflection w.r.t. its left edge. In the sequel we will work with the diagram $\Delta \cup \overline{\Delta}$. The boxes of this diagram will be also called superboxes. Similar to above, we will denote by l and r the left and the right shifts on the diagram $\Delta \cup \overline{\Delta}$.

Definition 4. *A (finite) sequence $\eta = \{b_0, \dots, b_n\}$ of superboxes of the diagram $\Delta \cup \overline{\Delta}$ is called an admissible path in this diagram, if the following two conditions hold:*

(1) *If $b_i \in \Delta$ then $b_{i+1} \in \{b_i, l(b_i)\}$;*
(2) *If $b_i \in \overline{\Delta}$ then $b_{i+1} \in \{b_i, l(b_i)\} \cup \Delta$*

(see an example on Figure 1). *The superboxes from the admissible path η will be called the* vertices *of the path. We will distinguish three types of vertices: the vertex b_m, $0 \le m < n$, will be called* walking, *if $b_{m+1} = l(b_m)$, it will be called* sleeping, *if $b_{m+1} = b_m$, and it will be called* jumping, *if $b_m \in \overline{\Delta}$ and $b_{m+1} \in \Delta$.*

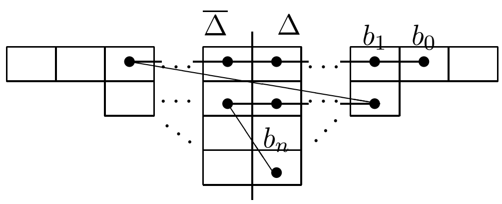

Figure 1.

Further, given any superbox x of $\Delta \cup \overline{\Delta}$ we will denote by $\bar x$ the superbox , obtained from x by the reflection of x w.r.t. the left edge of the diagram Δ. We also assume that the size of the superbox $x \in \overline{\Delta}$ is equal to the size of superbox $\bar x$.

From the definition of Darboux frame it follows that the quantity $-\omega(E_{a_i}, E_{a_j}^{(s+2)})$, we are interested in, is equal to the coefficient near F_{a_i} of the expansion of $E_{a_j}^{(s+2)}$ into linear combination w.r.t. the frame $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$, satisfying the structural equation (2.7). Admissible pathes in the diagram $\Delta \cup \overline{\Delta}$ help to describe the coefficients of such expansions. For this to any admissible path $\eta = \{b_0, \dots, b_n\}$ we will assign a curve of $\mathrm{size}(b_n) \times \mathrm{size}(b_0)$-matrices $P_\eta(\cdot)$. The curve of matrices $P_\eta(\cdot)$ can be defined by the recursive formulas on the number of vertices in η. If η consists of only one vertex, $\eta = \{b_0\}$, we set $P_\eta(t)$ to be the identity matrix for any t. Further for the path $\eta = \{b_0, \dots, b_{n-1}, b_n\}$ the curve of matrices $P_\eta(\cdot)$ is obtained from the curve of matrices $P_{\{b_0,\dots,b_{n-1}\}}$ by the following recursive formula:

$$P_{\{b_0,\dots,b_{n-1},b_n\}}(t) = \begin{cases} P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = l(b_{n-1}),\ b_{n-1} \in \Delta, \\ -P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = l(b_{n-1}),\ b_{n-1} \in \overline{\Delta}, \\ P'_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_n = b_{n-1}, \\ R_t(\bar b_{n-1}, b_n) P_{\{b_0,\dots,b_{n-1}\}}(t) & \text{if } b_{n-1} \in \overline{\Delta},\ b_n \in \Delta \end{cases} \tag{3.48}$$

Given $\{a, b\} \subset \Delta \cup \overline{\Delta}$ and $n \in \mathbb{N} \cup \{0\}$ denote by $\Omega(a, b, n)$ the set of all admissible pathes in the diagram $\Delta \cup \overline{\Delta}$, starting at a, ending at b, and consisting of $n+1$ vertices. Then from structural equation (2.7), definition (3.48) of matrices P_η, and elementary rules of differentiations it follows that

$$\omega(E_{a_i}, E_{a_j}^{(s+2)}) = - \sum_{\eta \in \Omega(a_j, \bar{a}_i, s+2)} P_\eta \tag{3.49}$$

Remark 2. It is clear from the last line of the recursive formula (3.48) that if $P_\eta(t) \neq 0$, then $R_t(\bar{b}_m, b_{m+1}) \neq 0$ for any jumping vertex b_m of η. $\square$

Further , it is convenient to enumerate the columns of the diagram $\Delta \cup \overline{\Delta}$ by integers in the following way: to the jth column (from the left) of Δ we assign the same number j while to the jth column from the right of $\overline{\Delta}$ we assign the number $1 - j$. Given a superbox $a \in \Delta \cup \bar{\Delta}$, denote by $c(a)$ the number of the column, according to the rule described in the previous sentence. The following simple lemma will be useful in the sequel

Lemma 7. *Suppose that $R_t : \Delta \times \Delta \mapsto \mathrm{Mat}$ is a quasi-normal mapping and $R_t(a, b) \neq 0$, where superboxes a and b lie in the jth and ith level of Δ respectively ($j < i$). Then the pair (a, b) is $\big(c(b) - c(\bar{a})\big)$th pair in the tuple (2.5).*

Indeed, by Definition 1 the nonzero matrix $R_t(a, b)$ must correspond to a pair from the appropriate tuple of the form (2.5). The second sentence of the lemma is obvious.

Proof of Statement 3. Fix some admissible path $\eta = \{b_0, \ldots, b_{s+2}\}$ from $\Omega(a_j, \bar{a}_i, s+2)$ (by definition, $b_0 = a_j$ and $b_{s+2} = \bar{a}_i$). Let us denote by k the number of jumping vertices in η. Further, let $b_{m_1}, \ldots, b_{m_k}$ be all jumping vertices of η, where $m_1 < m_2 < \ldots < m_k$. Set also $m_0 = -1$, $m_{k+1} = s+2$. It is evident that for any $1 \leq u \leq k+1$ the number of superboxes between $b_{m_{u-1}+1}$ and b_{m_u} (including $b_{m_{u-1}+1}$ but not b_{m_u}) is equal to $c\big(b_{m_{u-1}+1}\big) - c\big(b_{m_u}\big)$. Therefore the fact that all superboxes b_u with $0 \leq u < s+1$ are either walking or sleeping or jumping can be expressed as follows

$$\sum_{u=1}^{k+1} \big(c(b_{m_{u-1}+1}) - c(b_{m_u})\big) + \#\{\text{sleeping vertices of } \eta\} + k = s + 2. \tag{3.50}$$

Lemma 8. *Under assumptions of Statement 3 if $P_\eta \neq 0$ for a path $\eta \in \Omega(a_j, \bar{a}_i, s+2)$ ($j < i$) with $p_j - p_i \geq s$, then there is only one jumping vertex and there are no sleeping vertices in η.*

Proof. Since any path $\eta \in \Omega(a_j, \bar{a}_i, s+2)$ has to contain at least one jumping vertex (in order to jump somehow from jth to ith level) the lemma is actually equivalent to the fact that

$$\#\{\text{sleeping vertices of } \eta\} + k = 1 \tag{3.51}$$

Assume the converse, i.e.

$$\#\{\text{sleeping vertices of } \eta\} + k \geq 2. \tag{3.52}$$

Given a superbox $x \in \Delta$, denote by $p(x)$ the number of superboxes in the level of x. Assume that the superboxes b_{m_u} and b_{m_u+1} lie in different levels. By Remark 2, $R_t(\bar{b}_{m_u}, b_{m_u+1}) \neq 0$. Therefore, according to Lemma 7 either $(\bar{b}_{m_u}, b_{m_u+1})$ or $(b_{m_u+1}, \bar{b}_{m_u})$ is the $\big(c(b_{m_u+1}) - c(b_{m_u})\big)$th pair in the tuple (2.5). Combining this with Remark 2 and assumptions of Statement 3, one can obtain that if the superboxes b_{m_u} and b_{m_u+1} lie in different levels, then

$$c(b_{m_u+1}) - c(b_{m_u}) > \min\{s - 1, |p(b_{m_u+1}) - p(\bar{b}_{m_u})| - 1\}. \tag{3.53}$$

Further, since $c(b_0) = 1$ and $c(b_{s+2}) = 0$ (recall that $b_0 = a_j$, $b_{s+2} = a_i$, and $m_{k+1} = s+2$), we have

$$\sum_{u=1}^{k+1} \big(c(b_{m_{u-1}+1}) - c(b_{m_u})\big) = \sum_{u=1}^{k} \big(c(b_{m_u+1}) - c(b_{m_u})\big) + 1. \tag{3.54}$$

Substituting the last identity into (3.50) and using assumption (3.52) we obtain

$$\sum_{u=1}^{k} \big(c(b_{m_u+1}) - c(b_{m_u})\big) \le s - 1. \tag{3.55}$$

Since all terms in the sum in the lefthand side of the previous inequality are positive, we have $c(b_{m_u+1}) - c(b_{m_u}) \le s-1$ for any $1 \le u \le k$. Combining the last inequality with (3.53) we obtain that if the superboxes b_{m_u} and b_{m_u+1} lie in different levels, then

$$c(b_{m_u+1}) - c(b_{m_u}) \ge |p(b_{m_u+1}) - p(\bar{b}_{m_u})|. \tag{3.56}$$

Besides, if the superboxes b_{m_u} and b_{m_u+1} lie in the same level, then the inequality (3.56) holds automatically.

On the other hand, by our constructions the superboxes b_{m_u+1} and $\bar{b}_{m_{u+1}}$ lie in the same level of Δ. This fact together with inequalities (3.56) and (3.55) implies that

$$p_j - p_i \le \sum_{i=1}^{k} |p(b_{m_u+1}) - p(\bar{b}_{m_u})| \le \sum_{i=1}^{k} c(b_{m_u+1}) - c(b_{m_u}) \le s-1,$$

which contradicts the assumption $p_j - p_i \ge s$ of Lemma 8. The proof of the lemma is completed. $\Box$

Now, if η has only one jumping vertex and no sleeping vertices, then from (3.50) and (3.54) it follows that $c(b_{m_1+1}) - c(b_{m_1}) = s$. Besides, in this case the superbox b_{m_1} lies in the jth level and the superbox b_{m_1+1} lies in the ith level. But then from Remark 2 and Lemma 7 it follows that if $P_\eta \neq 0$ then the pair $(\bar{b}_{m_1}, b_{m_1+1})$ is exactly the sth pair of the tuple (2.5), which together with (3.48) and (3.49) implies (3.47). The proof of Statement 3 is completed. $\Box$

As we have already menstioned, Proposition 3 follows immediately from Statement 3 by induction w.r.t. s, starting with $s = 1$ (for which the assumptions of Statement 3 hold automatically).

3.7. **Final steps of the proof of Theorem 1.** The "if" part of Proposition 3 implies that the tuple $(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta})$ constructed in the subsection 3.5 is a normal moving frame of the curve $\Lambda(\cdot)$. Moreover, by the constructions of subsection 3.3 the space $V_i(t) = \mathrm{span}\{E_{\sigma_i}(t)\}$ is the canonical complement of $\big(\Lambda_{(p_i)}\big)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ for any $1 \le i \le d$, where σ_i is the special superbox of the ith level, and by constructions of subsection 3.4 the curves $E_{\sigma_i}(t)$ are horizontal sections of the bundle $\mathfrak{B}_i$, defined in subsection 3.3.

Now suppose that $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ is another normal moving frame of the curve $\Lambda(\cdot)$. From the second line of the structural equation (2.7) (where all $E_a(t)$ and $F_a(t)$ are replaced by $\widetilde{E}_a(t)$ and $\widetilde{F}_a(t)$) and the definition of Darboux frame it follows that conditions (3.29) (again with all $E_a(t)$ replaced by $\widetilde{E}_a(t)$) hold for any $1 \le j < i \le d$ and $k = 1, 2$. Indeed, $\omega\big(\widetilde{E}_{a_i}(t), \widetilde{E}'_{a_j}(t)\big) = \omega\big(\widetilde{E}_{a_i}(t), \widetilde{F}_{a_j}(t)\big) = 0$ and $\omega\big(\widetilde{E}_{a_i}(t), \widetilde{E}''_{a_j}(t)\big) = -\omega\big(\widetilde{E}'_{a_i}(t), \widetilde{E}'_{a_j}(t)\big) = -\omega\big(\widetilde{F}_{a_i}(t), \widetilde{F}_{a_j}(t)\big) = 0$. Further, by Proposition 3, from the normality of the frame $(\{\widetilde{E}_a(t)\}_{a\in\Delta}, \{\widetilde{F}_a(t)\}_{a\in\Delta})$ it follows that conditions (3.29) (again with all $E_a(t)$ replaced by $\widetilde{E}_a(t)$) hold for any $1 \le j < i \le d$ and $3 \le k \le p_j - p_i + 1$. Therefore, Lemma 5 implies that $\mathrm{span}\{\widetilde{E}_{\sigma_i}(t)\} = \mathrm{span}\{E_{\sigma_i}(t)\} = V_i(t)$.

Besides, from the second line of the structural equation (2.7) (where again all $E_a(t)$ and $F_a(t)$ are replaced by $\widetilde{E}_a(t)$ and $\widetilde{F}_a(t)$) and Proposition 1 it follows that the curves $\widetilde{E}_{\sigma_i}$ are horizontal sections of the bundle $\mathfrak{B}_i$, which together with (3.33) implies relations (2.8). This completes the proof of Theorem 1.

4. Nonmonotonic curves satisfying condition (G)

Now consider not necessarily monotonic curves with fixed Young diagram D and reduced Young diagram Δ, satisfying condition (G) (see subsection 3.3). For such curves the canonical complements $V_i(t)$ to $\left(\Lambda_{(p_i)}\right)^{(1)}(t)$ in $\Lambda_{(p_i-1)}(t)$ are defined as well. Denote by Γ_i^+ and Γ_i^- the positive and the negative index of the quadratic form $\dot\Lambda(t)|_{(\Lambda_{(p_i-1)})^{(p_i-1)}(t)}$ and let $r_i^+ = \Gamma_i^+ - \Gamma_{i-1}^+$ and $r_i^- = \Gamma_i^- - \Gamma_{i-1}^-$. Actually the numbers r_i^+ and r_i^- are equal to the positive and negative inertia index of the canonical quadratic forms $Q_{i,t}$ on $V_i(t)$. These numbers do not depend on t and they will be called *the ith positive inertia index and the ith negative inertia index of the curve* $\Lambda(t)$ respectively. Similarly to Definition 3 one can define the normal (quasi-normal) moving frame for a curve in a Lagrange Grassmannian, satisfying condition (G). The only modification comparing to this definition is that one should replace the second line in the structural equation (2.7) by $E_a' = I_{r_i^+,r_i^-}F_a(t)$, $a \in \mathcal{F}_1 \cap \Upsilon_i$, where r_i^+ and r_i^- are the ith positive and negative inertia indices of the curve $\Lambda(t)$, and the matrix $I_{r_i^+,r_i^-}$ is the diagonal $(r_i^+ + r_i^-) \times (r_i^+ + r_i^-)-$matrix such that its first r_i^+ diagonal entries are equal to 1 and others are equal to -1. Continuing the normalization procedure by complete analogy with subsections 3.4-3.6 with obvious modifications, one gets the following generalization of Theorem 1 to nonmonotonic curves satisfying condition (G):

Theorem 3. *For any curve $\Lambda(t)$ with the Young diagram D in the Lagrange Grassmannian, satisfying condition (G), there exists a normal moving frame $\big(\{E_a(t)\}_{a\in\Delta}, \{F_a(t)\}_{a\in\Delta}\big)$. A moving frame $\big(\{\tilde e_\alpha(t)\}_{\alpha\in D}, \{\tilde f_\alpha(t)\}_{\alpha\in D}\big)$ is a normal moving frame of the curve $\Lambda(\cdot)$ if and only if for any $1 \le i \le d$ there exists a constant matrix $U_i \in O(r_i^+, r_i^-)$ such that for all t*

$$\widetilde{E}_a(t) = E_a(t)U_i, \quad \widetilde{F}_a(t) = F_a(t)I_{r_i^+,r_i^-}U_iI_{r_i^+,r_i^-}, \quad \forall a \in \Upsilon_i, \tag{4.1}$$

where r_i^+ and r_i^- are the ith positive and the negative inertia indices of the curve $\Lambda(t)$.

Further, take a Young diagram D, as before, and fix a tuple of nonnegative integers $\{r_i^-\}_{i=1}^d$ such that $0 \le r_i^- \le r_i$ for any $1 \le i \le d$. Let $\mathfrak{Q}_D$ be the quiver, defined in subsection 2.3. A representation of the quiver $\mathfrak{Q}_D$ will be called *compatible with the Young diagram D and the tuple $\{r_i^-\}_{i=1}^d$*, if for any $1 \le i \le d$ the space of the representation corresponding to the vertex Υ_i is a r_i-dimensional pseudo-Euclidean space with negative inertia index r_i^- and the linear mappings $\mathcal{R}(a,b)$ of the representation corresponding to arrows (a,b) satisfy the following relations: $\mathcal{R}(a,b)^* = \mathcal{R}(b,a)$ and $\mathcal{R}\big(a, r(a)\big)$ is antisymmetric w.r.t. the corresponding pseudo-Euclidean structure. Then by complete analogy with Theorem 2 we have

Theorem 4. *For the given one-parametric family $\Xi(t)$ of representations of the quiver $\mathfrak{Q}_D$ compatible with the Young diagram D with $|D|$ boxes and the tuple of nonnegative integers $\{r_i^-\}_{i=1}^d$ there exists the unique, up to a symplectic transformation, curve $\Lambda(t)$, satisfying condition (G), in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that the quiver of curvatures of $\Lambda(t)$ is isomorphic to $\Xi(t)$ and its ith negative inertia index is equal to r_i^- for any $1 \le i \le d$. If, in addition, all rows of D have different length, then given a tuple of smooth functions $\{\rho_{a,b}(t) : (a,b) \in \Delta \times \Delta, (a,b)$ is an essential pair$\}$ there exists the unique, up to a symplectic transformation, curve $\Lambda(t)$, satisfying condition (G), in the Lagrange Grassmannian of $2|D|$-dimensional symplectic space with the Young diagram D such that for* *any essential pair $(a, b) \in \Delta \times \Delta$ and any t its (a, b)-curvature at t coincides with $\rho_{a,b}(t)$ and its ith negative inertia index is equal to r_i^- for any $1 \le i \le d$.*

5. Consequences for differential geometry of geometric structures on manifolds

Let $\mathcal{V}$ be a geometric structure on a manifold M, as in the Introduction, and H be its maximized Hamiltonian, which is smooth on an open subset of T^*M. Assume that the point $\lambda \in T^*M$ satisfies: $H(\lambda) > 0$, $dH(\lambda) \neq 0$, and the germ of the Jacobi curve $J_\lambda(t)$ at $t = 0$ has Young diagram D with the reduced diagram Δ and with p_1 boxes in the first row. Let, as before, $W_\lambda = T_\lambda \mathcal{H}_{H(\lambda)}/\{\mathbb{R}\vec{H}(\lambda)\}$ be the symplectic space, where the Jacobi curve $J_\lambda(t)$ lives. The point λ will be called *D-regular* if, in addition to above,

$$J_\lambda^{(p_1)}(0) = W_\lambda \tag{5.1}$$

and the germ of the Jacobi curve $J_\lambda(t)$ at $t = 0$ satisfies condition (G). Here for simplicity we will work mainly with D-regular points for some Young diagram D. Let

$$J_\lambda(0) = \oplus_{a\in\Delta}\widetilde{\mathfrak{A}}_a(\lambda) \tag{5.2}$$

be the canonical splitting of the subspace $J_\lambda(0)$ (w.r.t. the canonically parameterized curve $J_\lambda(0)$) and $\mathrm{proj}_\lambda : T_\lambda\mathcal{H}_{H(\lambda)} \mapsto W_\lambda$ be the canonical projection on the factor-space. Set

$$\mathfrak{A}_a(\lambda) = (\mathrm{proj}_\lambda)^{-1}\big(\widetilde{\mathfrak{A}}_a(\lambda)\big) \cap \Pi_\lambda, \tag{5.3}$$

where Π_λ is the vertical subspace of $T_\lambda\mathcal{H}_{H(\lambda)}$, defined by (1.2). Taking into account that proj_λ establishes an isomorphism between Π_λ and $J_\lambda(0)$, we get from (5.2) and (5.3) the following *canonical splitting of the tangent space $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$ to the fiber of T^*M at λ*:

$$T_\lambda T^*_{\pi(\lambda)}M = \oplus_{a\in\Delta}\widetilde{\mathfrak{A}}_a(\lambda) \oplus \mathrm{span}\,\{\epsilon(\lambda)\}, \tag{5.4}$$

where ϵ is the Euler field of T^*M, i.e. the infinitesimal generator of the homotheties of the fibers of T^*M. Besides, each subspace $\mathfrak{A}_a(\lambda)$ is endowed with the canonical pseudo-Euclidean structure and the corresponding curvature mappings between the subspaces of the splitting are intrinsically related to the geometric structure $\mathcal{V}$.

Further, let

$$\mathrm{Hor}(\lambda) = (\mathrm{proj}_\lambda)^{-1}\big(J_\lambda^{\mathrm{trans}}(0)\big), \tag{5.5}$$

where $J_\lambda^{\mathrm{trans}}(0)$ is the subspace corresponding to the canonical complementary curve to the Jacobi curve J_γ at $t = 0$. Then $\mathrm{Hor}(\lambda)$ is transversal to the tangent space $T_\lambda\big(T^*_{\pi(\lambda)}M\big)$ to the fiber of T^*M at λ. So, if for some diagram D the set U of its regular D-points is open in $T^*M\backslash H^0$, then for any $q \in \pi(U)$ the subsets $T_q^*M \cap U$ of the linear space T_q^*M is endowed with very rich additional structures: at each point $\lambda \in T_q^*M \cap U$ there is the canonical splitting of tangent spaces (smoothly depending on λ) such that the subspaces of the splitting are parameterized by the superboxes of the reduced diagram Δ, the dimension of each subspace is equal to the size of the corresponding superbox, these subspaces are endowed with the canonical pseudo-Euclidean structures, and the canonical linear mappings between these subspaces (i.e. the (a, b)-curvature mappings) are defined. Besides the distribution of "horizontal" subspaces $\mathrm{Hor}(\lambda)$ defines the *connection on $U \subset T^*M$, canonically associated with geometric structure $\mathcal{V}$.*

In the case of sub-Riemannian structures the Hamiltonian H^2 is nonnegative quadratic form on the fibers. First it implies the monotonicity of the corresponding Jacobi curves. Further assume that in this case relation (5.1) holds for some λ and p_1. Then there is a neighborhood U of $\pi(\lambda)$ in M and an open and dense subset $\mathcal{O}$ of U that satisfies the following property: for any $\tilde{q} \in \mathcal{O}$ there exists a neighborhood $\widetilde{U} \in \mathcal{O}$ and a Young diagram D such that for each $\hat{q} \in \widetilde{U}$ the intersection of the set of its D-regular points with $T^*_{\hat{q}}M$ is an nonempty Zariski open subset of $T^*_{\hat q}M$. Besides, if one works with the Hamiltonian H^2 instead of H, the canonical splitting, the canonical Euclidean structures on the subspaces of the splitting, the curvature mappings, and the canonical connection above depend rationally on points of the fibers of T^*M. So, to any sub-Riemannian metric satisfying assumptions above one can assign very rigid additional structures on T^*M.

Condition (5.1) has the following equivalent description in terms of the extremal $e^{t\vec H}\lambda$. Projections of the Pontryagin extremals to the base manifold M are called extremal trajectories. Conversely, an extremal projected to the given extremal trajectory is called its *lift.* From the Pontryagin Maximum Principle it follows that the set of all lifts of given extremal trajectory can be provided with the structure of linear space. The dimension of this space is called *corank of the extremal trajectory.* It turns out that if condition (5.1) holds, then corank of the extremal trajectory $\pi(e^{t\vec H}\lambda)$ is equal to 1. Conversely, if corank of the extremal trajectory $\pi(e^{t\vec H}\lambda)$ is equal to 1, then $J^{(p_1(t))}_{e^{tH}\lambda}(0) = W_{e^{tH}\lambda}$ for t from generic set. Note also that if corank of the extremal trajectory is greater than 1, then this extremal trajectory is the projection of a so-called abnormal extremal (a Pontryagin extremal living on zero level set of the corresponding Hamiltonian).

Conjecture *(private communication with Andrei Agrachev and Tohru Morimoto). Any sub-Riemannian metric on a completely nonholonomic vector distribution has at least one corank 1 extremal trajectory or, equivalently, not all extremal trajectories of it are projections of abnormal extremals.*

If the conjecture is true, then the construction above can be implemented for any sub-Riemannian metric on a completely nonholonomic vector distribution.

In the case of a Riemannian metric the canonical connection above coincides with the Levi-Civita connection, the reduced Young diagram of Jacobi curves consists of only one superbox, and the corresponding curvature mapping can be identified with the part of Riemannian curvature tensor appearing in the classical Jacobi equation for Jacobi vector fields along the Riemannian geodesics([1]). In particular, the whole Riemannian curvature tensor can be recovered from it. In general case the relation between (a, b)-curvature mappings and the curvature tensor of the canonical connection is subject for further study.

Finally, if the Jacobi curve $J_\lambda(t)$ has Young diagram D with p_1 boxes in the first row such that $J^{(p_1)}_\lambda(t) \subsetneq W_\lambda$, then using Remark 1, one can make analogous construction on the space $J^{(p_1)}_\lambda(0)/(J^{(p_1)}_\lambda(0))^\angle$.

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