Math 251
Test I Fall 2005 (Sept. 30, 2005)
Section 508 (8–9:15 am) 509 (9:35–10:50 am)

There are a total of 4 problems. No calculators are allowed.

1. Let \( n \) be a positive integer greater than 2, and

\[
f(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{n}{2}}.
\]

(a) Show that \( f \) satisfies the Laplace equation

\[
\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{for} \quad (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0).
\]

(b) Evaluate \( \frac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_2^2 \partial x_2} \).

(c) If \( n = 2 \) in (*), determine the value of \( f(1, 3) \).

1. (a) Exactly the same as before.

(b) \[
\frac{\partial f}{\partial x_1} = (2-n) x_1 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}}
\]

\[
\frac{\partial^2 f}{\partial x_1^2} = (2-n) x_1 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2} - 1} \left( -\frac{n}{2} \right) \frac{\partial}{\partial x_1} (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}}
\]

\[
= -(2-n) n x_1 x_3 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2} - 1}
\]

(c) \[
\nabla (x_1, x_2) = \left( 1^2 + 2^2 \right)^{\frac{2-n}{2}} = |\delta^0| = 1.
\]
2. Let $a > 0$ be a given constant. A quadric surface is given as
\[ ax^2 + y^2 - az = 0. \]

(a) Find the tangent plane to the surface at the point $(a, a, a + a^2)$. 
(b) Use differentials to compute an approximate value for
\[ \sqrt[3]{6(9.994)^2 + 25(15.91)} \]
with 4 decimal place accuracy.

(a) \[ \nabla \Phi = (2x, \frac{2}{a}y) = (2a, \frac{2}{a}a) = (2a, 2) \] when $(x, y) = (a, a)$.

So the eq'n for the tangent plane is
\[ z - (a + a^2) = (2a, 2) \cdot (x - a, y - a) = 2ax - 2a + 2y - 2a, \]
i.e.
\[ 2ax + 2y - z = a^2 + a. \]

(b) \[
\begin{align*}
\Delta x &= 9.994 - 10 = -0.006 \\
\Delta y &= 15.91 - 16 = -0.9 \\
\end{align*}
\]
\[ f(x, y) = \sqrt[3]{6x^2 + 25y} = \sqrt[3]{6(10)^2 + 25(16)} = \sqrt[3]{600 + 400} = 10, \text{ at } (x, y) = (10, 16). \]

\[ f(9.994, 15.91) \approx f(10, 16) + \frac{\partial f}{\partial x}(10, 16) \Delta x + \frac{\partial f}{\partial y}(10, 16) \Delta y \]
\[ \frac{\partial f}{\partial x} = \frac{1}{3} (6x^2 + 25y)^{-\frac{2}{3}} \cdot 12x \quad \text{at } (x, y) = (10, 16) \Rightarrow \left\{ \begin{align*}
\frac{\partial f}{\partial x} &= \frac{1}{3} \frac{1}{10^2} \cdot 12 \cdot 10 \\
\frac{\partial f}{\partial y} &= \frac{1}{3} \frac{1}{10^2} \cdot 25
\end{align*} \right. \]

Hence
\[ f(9.994, 15.91) \approx f(10, 16) + \frac{1}{3} \cdot \frac{1}{10^2} \cdot 12(10) \cdot (-0.006) + \frac{1}{3} \cdot \frac{1}{10^2} \cdot 25(-0.09) \]
\[ = 10 - 0.0024 - 0.0075 = 10 - 0.0099 \]
\[ = 9.9901. \]
3. Given two lines

\[
\begin{align*}
L_1: & \quad x = 2s, \quad y = 3 + 4s, \quad z = -3 + s, \\
L_2: & \quad x = 2 - t, \quad y = -1 + t, \quad z = 5 - 3t,
\end{align*}
\]
\(s, t \in \mathbb{R},\)

(a) show that \(L_1\) and \(L_2\) are skew lines;

(b) compute the distance between \(L_1\) and \(L_2\).

(a) First, note that the directions of \(L_1\) and \(L_2\) are, resp., \((2, 4, 1)\) and \((-1, 1, -3)\).

These two directions are not proportional. Hence \(L_1\) and \(L_2\) are not parallel.

Next, we determine if \(L_1\) and \(L_2\) intersect. Let

\[
\begin{align*}
2s = 2 - t & \quad (x 2): \quad 4s = 4 - 2t \\
3s + 4s = -1 + t & \quad \rightarrow 3s + 4s = -1 + t \\
-3 = 5 - 3t & \quad \Rightarrow 3t = 8, \quad t = 8/3.
\end{align*}
\]

Substituting \(s = -\frac{1}{3}\), \(t = 8/3\) into \(\frac{y}{-3} + \frac{x}{1} + \frac{z}{-3} = 0\), we obtain

\(-3 + \frac{-1}{3} = 5 - 3(\frac{8}{3})\) \(\Rightarrow \frac{-10}{3} - 3\), a contradiction. So \(L_1\) and \(L_2\) are skew lines.

(b) First, find a common direction \(\vec{n}\) which is perpendicular to both lines:

\[
\vec{n} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 4 & 1 \\
-1 & 1 & -3
\end{vmatrix} = (12 - 1) \hat{i} + (6 - 1) \hat{j} + (2 + 4) \hat{k} = 11 \hat{i} + 5 \hat{j} + 6 \hat{k}
\]

Find a plane passing \((0, 3, -3)\) with normal \(\vec{n}\) (so that this plane contains \(L_1\)):

\[-13(x - 0) + 5(y - 3) + 6(z + 3) = 0,
\]

i.e.,

\[-13x + 5y + 6z = -3.
\]

Find a second plane passing \((2, -1, 5)\) with normal \(\vec{n}\) (so that this plane contains \(L_2\) and is parallel to the first plane):

\[-13(x - 2) + 5(y + 1) + 6(z - 5) = 0,
\]

i.e.,

\[-13x + 5y + 6z = -1.
\]

The distance between \(L_1\) and \(L_2\) is the distance between the two planes.

Thus

\[
d = \frac{|1-3|}{\sqrt{(-13)^2 + 5^2 + 6^2}} = \frac{2}{\sqrt{230}}.
\]
4. (a) Let

\[ f(x, y) = \frac{-7x^2y}{2x^4 + 3y^2}, \text{ if } (x, y) \neq (0, 0). \]

Show that \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist. \((10\%)\)

(b) Let

\[ f(x, y) = \frac{-7x^4y^2}{9x^4 + 5y^2} + 3, \text{ if } (x, y) \neq (0, 0). \]

Prove that \( \lim_{(x,y) \to (0,0)} f(x, y) = 3. \)

(You must check the definition for a limit rigorously by an \((\varepsilon, \delta)\)-argument.) \((13\%)\)

(a) Along the line \( y = x \), we have

\[
\lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{-7x^3 \cdot x}{2x^4 + 3x^4} = \lim_{x \to 0} \frac{-7x^3}{2x^4 + 3x^4} = \lim_{x \to 0} \frac{-7x}{2x^2 + 3} = \frac{0}{3} = 0.
\]

Along the curve \( y = x^2 \), we have

\[
\lim_{x \to 0} f(x, x^2) = \lim_{x \to 0} \frac{-7x^3 \cdot x^2}{2x^4 + 3x^4} = \lim_{x \to 0} \frac{-7x^3}{2x^4 + 3x^4} = \frac{-7}{5}. \]

Since 0 \( \neq \frac{-7}{5} \), from the uniqueness of a limit if it exists, we conclude that the limit doesn't exist.

(b) Given any \( \varepsilon > 0 \), we want to show that we can choose \( \delta > 0 \) such that

\[
|f(x, y) - 3| < \varepsilon \quad \text{if} \quad 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta.
\]

\[
|f(x, y) - 3| = \left| -\frac{7x^4y^2}{9x^4 + 5y^2} \right| = \left| -\frac{9x^4}{9x^4 + 5y^2} \right| \left| \frac{7}{9}y^2 \right|
\]

\[
\leq \left| \frac{7}{9}y^2 \right| = \frac{7}{9}y^2 \leq \frac{7}{9} \left( \sqrt{x^2 + y^2} \right)^2 < \frac{7}{9} \delta^2 = \varepsilon,
\]

if we choose \( \delta = \frac{3}{\sqrt{7}} \varepsilon \). Hence we have proved that the limit exists.