There are a total of eight problems. No calculators are allowed.

1. (a) Use Stokes' Theorem to compute the integral \( \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x,y,z) = \begin{pmatrix} yz \mathbf{i} - xz \mathbf{j} + (x^2-xy+y^2) \mathbf{k} \end{pmatrix} \) and the surface \( S \) is the part of the sphere \( x^2+y^2+z^2 = a^2 \) that lies inside the cylinder \( y^2 + z^2 = b^2 \) and satisfies \( x > 0 \). Note that \( a > b > 0 \).

(b) Give a sketch of the surface and region under consideration in (a).

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**Solution**

Stokes' Theorem gives \( \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \), where \( S \) and \( \partial S \) can be seen from the sketch below:

On \( \partial S \), \( x^2+y^2+z^2 = x^2+(y^2+z^2) = x^2+b^2 = a^2 \), so \( x = \sqrt{a^2-b^2} \) (\( > 0 \)). The curve \( \partial S \) lies on \( y^2+z^2 = b^2 \). So we parametrize it by

\[
\begin{align*}
  y &= b \cos \theta \\
  z &= b \sin \theta \\
  0 &\leq \theta \leq 2\pi,
\end{align*}
\]

\[
\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( (y^2)dx + (-x^2)dy + (b^2-x^2+y^2)dz \right)
\]

\[
= \int_0^{2\pi} \left( b \cos \theta \cdot b \sin \theta \cdot d(\sqrt{a^2-b^2}) + \sqrt{a^2-b^2} \cdot b \sin \theta \cdot d(b \cos \theta) + (b^3 \sin^2 \theta - \sqrt{a^2-b^2} \cdot b \cos \theta + b \cos \theta) \cdot d(b \sin \theta) \right)
\]

\[
= \int_0^{2\pi} \left( 0 + b^2 \sqrt{a^2-b^2} \sin \theta \cdot d\theta + (b \sqrt{a^2-b^2} \cos \theta - b \sqrt{a^2-b^2} \cos \theta + b^2 \cos \theta) \cdot d\theta \right)
\]

\[
= \pi b^2 \sqrt{a^2-b^2} + b^2 \left( 1 - \sqrt{a^2-b^2} \right) \pi = \pi b^2.
\]
2. Selected parts from
   (a) Exercise 20, p. 931 and
   (b) Exercises 21–26, p. 937.

3. Find the flux of the vector field \( \mathbf{E}(x, y, z) = (5x^2 - y) \mathbf{i} + 6y \mathbf{j} - (x-63) \mathbf{k} \) over the sphere \( (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \), where \( r > 0 \).

   (Gauss's Divergence Theorem)

   **Solu** Flux \( \iiint_{S} \mathbf{E} \cdot d\mathbf{S} = \iiint_{V} \nabla \cdot \mathbf{E} \, dV. \)

   \[ \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (5x^2 - y) + \frac{\partial}{\partial y} (6y) + \frac{\partial}{\partial z} (x-63) = 10x + 6 + 6 = 10x + 12 \]

   The ball \( V \) can be parameterized by

   \[
   \begin{align*}
   x &= a + r \sin \phi \cos \theta, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi, \\
   y &= b + r \sin \phi \sin \theta, \\
   z &= c + r \cos \phi, \quad 0 \leq r \leq r
   \end{align*}
   \]

   

   So

   \[
   \iiint_{V} (10x + 12) \, dV = \iiint_{V} [10(x-a^2 + b^2 + c^2)] \, dV
   \]

   \[
   = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{r} 10 \rho^2 \sin \phi \cos \theta + 10 \rho^2 \sin \phi \sin \theta + 10 \rho^2 \cos \theta \, d\rho \, d\phi \, d\theta
   \]

   \[
   = 10 \int_{\theta=0}^{2\pi} \cos \theta \, d\theta \int_{\phi=0}^{\pi} \sin^2 \phi \, d\phi \int_{\rho=0}^{r} \rho^2 \, d\rho + (10a^2 + 12) \iiint_{V} dV
   \]

   \[
   = (10a^2 + 12) \cdot \text{Volume of the ball}
   \]

   \[
   = (10a^2 + 12) \cdot \frac{4}{3} \pi r^3.
   \]
4. Exercise 23, p.899. (Greens Theorem) (Already discussed in class).

5. To be selected from Test I, Problems 1 and 4:
   Problem 1 is the Laplace equation;
   Problem 4 is the limit.
   See the Answer Key posted.

6. To be selected from Test II, Problems 1 and 2:
   Problem 1 is the constrained max and min problem;
   Problem 2 is the critical point problem.
   See the Answer Key posted.

7. Problem 4, Test II:
   \[ \int_{-2}^{2} \int_{-2}^{2} \sin(-x^2 + 12x + 5) \, dx \, dy \]

8. To be selected from Test III, Problems 1 and 5:
   Problem 1 involves finding \( f \) such that \( \vec{F} = \nabla f \);
   Problem 5 involves finding the volume of an ice cream cone.
   See the Answer Key posted.