Complex Analysis
Qualifying Examination

January 2024

In this problem set, $\mathbb{D}$ is the open unit disk centered at zero.

1. (a) State Harnack's Inequality; Riemann Mapping Theorem; Great Picard Theorem on essential singularities.

(b) Sketch the proof of one of these theorems.

2. Show that
$$\int_0^\infty \frac{\cos\sqrt{x}}{\sqrt{x}(1+x)}dx = \frac{\pi}{e}.$$

3. For each n, find the imaginary part of the integral $\int_\gamma f(z)dz$ where
$$f(z) = \frac{e^z}{z(z+1)}$$
and γ is a spiral given by $\gamma(t) = e^{-t+it}$, $t \in [0, 2\pi n]$.

4. An entire function f satisfies $|f(z)| \leq 1 + \sqrt{|z|}$. Prove that it is constant.

5. The non-constant function f is analytic in a neighborhood of the unit disc. It does not take purely imaginary values on the unit circle. Prove that on $\mathbb{D}$, we have $f = e^g$, where g is analytic in $\mathbb{D}$.

6. The function f is meromorphic and non-constant on a connected neighborhood of $\overline{\mathbb{D}}$, with poles $a_1, \ldots, a_n$ and zeros $b_1, \ldots, b_m$ (counted with mulitplicities), all located strictly inside $\mathbb{D}$.

Meromorphic functions f_k converge to f as $k \to \infty$, uniformly on compact subsets of $\mathbb{D} \setminus \{a_1, \ldots, a_n\}$.

(a) Show that for sufficiently large k, the number of zeros minus the number of poles of f_k in $\mathbb{D}$ (counting with multiplicities) is $m - n$.

(b) Is it always true that for sufficiently large k, the function f_k has m zeros in $\mathbb{D}$ (counting with multiplicities)?

7. Consider the family $\mathcal{A}$ of functions $f\colon \mathbb{D} \to \mathbb{C}$ that are holomorphic in $\mathbb{D}$ and satisfy (1) $f(0) = 0$; (2) for any $z \in \mathbb{D}$, $|\operatorname{Re} f(z)| < 1$.

(a) Show that there exists a real number M such that for all $f \in \mathcal{A}$, we have $|f'(0)| < M$.

(b) Show that this family is compact: any sequence $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ has a subsequence that converges locally uniformly on $\mathbb{D}$ to a function from $\mathcal{A}$.

8. Two circles ω_1, ω_2 are disjoint and located outside of each other. Let $\mathcal{C}$ be the family of circles that are externally tangent to both ω_1 and ω_2. Show that there exists a circle ω_3 such that all circles in the family $\mathcal{C}$ are perpendicular to ω_3.

9. For any two non-constant, non-proportional entire functions f, g, show that there exists a point $z \in \mathbb{C}$ such that

$$f^4(z) + f^3(z)g(z) + f^2(z)g^2(z) + f(z)g^3(z) + g^4(z) = 0.$$

10. The function f is meromorphic on $\mathbb{C}$ and has no poles on the real line.

(a) Prove that there exist two meromorphic functions $g_1, g_2 \colon \mathbb{C} \to \mathbb{C}$ such that $f = g_1 g_2$, the poles of g_1 are all in the upper half-plane, and the poles of g_2 are all in the lower half-plane.

(b) Prove that there exist two meromoprhic functions $h_1, h_2 \colon \mathbb{C} \to \mathbb{C}$ such that $f = h_1 + h_2$, the poles of h_1 are all in the upper half-plane, and the poles of h_2 are all in the lower half-plane.