Algebra Qualifying Examination Problems
August 2018

Instructions:

- Read all 9 problems first; make sure that you understand them and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. Do 'scratch work' on a separate page.
- Start each problem on a new page, clearly marking the problem number and your name on that page.
- Rings always have an identity and all R-modules are left modules.

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1. Let G be a finite group. Prove that the number of ordered pairs $(g, h) \in G^2$ such that g and h commute is equal to $k|G|$, where k is the number of conjugacy classes in G.

2. Here S_n and A_n are the symmetric and the alternating groups on n objects. You may use the fact that A_n is simple for $n \geq 5$ and that if H is a simple subgroup of S_n of order more than 2, then $H \subseteq A_n$.

   (a) Show that every homomorphism $A_6 \longrightarrow S_4$ is trivial

   (b) Show that A_6 has no subgroups of index 4.

   (c) Let G be a group of order 90 with no normal subgroups of order 5. Show that there is a non-trivial homomorphism $G \longrightarrow S_6$. (Hint: consider the Sylow 5-subgroups.)

   (d) Show that there are no simple groups of order 90.

3. Let $R = \mathbb{C}[x, y]/(x^3, y^3)$.

   (a) Find all prime ideals of R.

   (b) Show that R has a unique maximal ideal.

   (c) Find all units of R.

4. Show that the ideal $I = (3, x^6 + 1)$ is not a prime ideal of $\mathbb{Z}[x]$. Find prime ideals $A \neq 0$ and B such that $A \subset I \subset B \subset \mathbb{Z}[x]$.

5. Let R be a domain and F be its field of fractions. Prove that F is an injective R-module.

6. (a) Prove that every element of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written in the form $x \otimes 1$ for $x \in \mathbb{Q}$. (b) Prove that the map $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ generated by $a \otimes b \mapsto ab$ is an isomorphism of additive groups.

7. Let A, B, C be R-modules, where R is commutative with 1. Suppose that there is an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

(a) Show that if A and C are free R-modules, then B is a free R-module. (b) Prove that if an ideal I of R is a free R-module, then I is principal. (c) Suppose that R is not a PID. Show that there is an exact sequence as in part (a) where B is free but neither A nor C is free.

8. Let K be a field of characteristic 0 such that every odd degree polynomial $f(x) \in K[x]$ has a root in K. Let L/K be a finite extension. Show that $[L : K]$ is a power of 2.

9. Find the Galois group of the splitting field of $x^4 - 3$ over $\mathbb{Q}[\sqrt{-1}]$.