Qualifying Exam – January 12, 2023

1. Let $f(x) = x^3 + 2x + 2$ and $g(x) = x^3 + x + 1$, and let K be the splitting field of g. Does f have a root in K? (You may use without proof the fact that the discriminant of a cubic of the form $x^3 + ax + b$ is $-4a^3 - 27b^2$.)

2. Let R be a commutative ring with multiplicative identity 1_R.

   (a) If M_1 and M_2 are noetherian R-modules, then $M_1 \oplus M_2$ is also noetherian.

   (b) If R is noetherian and N is a finitely generated R-module, then N is noetherian.

3. Let F be a field, $V \neq \{0\}$ an F-vector space, and B a symmetric bilinear form on V. (That is, $B\colon V \times V \to F$ is F-linear in each argument, and $B(u, v) = B(v, u)$ for all $u, v \in V$.) Assume that $\mathrm{char}(F) \neq 2$ and that $1 \leq \dim_F(V) < \infty$.

   (a) Prove that if $B \neq 0$, then there exists $v \in V$ with $B(v, v) \neq 0$.

   (b) Suppose that $v \in V$ with $B(v, v) \neq 0$, and set $W = \{w \in V \mid B(w, v) = 0\}$. You may assume that W is a subspace of V. Show that $V = Fv \oplus W$.

   (c) Using parts (a) and (b), conclude in 1 or 2 sentences that there is a basis $\{v_t\}$ of V such that $B(v_i, v_j) = 0$ for all $i \neq j$.

4. Show that every group of order 33 is abelian.

5. Consider the subring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Is the principal ideal (11) a prime ideal of R?

6. Show that an abelian group of order 18 cannot act faithfully on a set of size 7. (Recall that the action of a group G on a set S is called *faithful* if the only element g of G satisfying $gs = s$ for all $s \in S$ is the identity element of G.)

7. Let R be a commutative ring with multiplicative identity 1_R. For $r \in R$, let
$$S_r = \{1_R\} \cup \{r^k \mid k \geq 1\},$$
and consider the localization map $\varphi_r : R \to S_r^{-1}R$.

   (a) For which $r \in R$ is φ_r injective?

   (b) For which $r \in R$ is $\varphi_r(a) = 0$ for all $a \in R$?

8. Let $\mathrm{GL}_3(\mathbb{Z}_3)$ be the group of invertible 3×3 matrices with entries in the field with 3 elements. Suppose that B is the subgroup of $\mathrm{GL}_3(\mathbb{Z}_3)$ of (weakly) upper triangular matrices and that
$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z}_3 \right\}$$
is the subgroup of B with 1's along the main diagonal.

   (a) Show that N is a normal subgroup of B.

   (b) What is the order of the quotient group B/N?

   (c) Is N normal in $\mathrm{GL}_3(\mathbb{Z}_3)$?