Applied/Numerical Analysis Qualifying Exam

August 13, 2010

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Part 1: Applied Analysis

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

1. Let $\mathcal{H}$ be a complex (separable) Hilbert space, with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ being the inner product and norm.

   (a) Define the term *compact linear operator* on $\mathcal{H}$.

   (b) Let $K : \mathcal{H} \to \mathcal{H}$ be compact. Show: *If $\lambda \neq 0$ is an eigenvalue of K, then it has finite multiplicity.*

2. Let $\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx$, where $w \in C[-1,1]$, $w(x) > 0$, and $w(-x) = w(x)$. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be the orthogonal polynomials generated by using the Gram-Schmidt process on $\{1, x, x^2 \ldots\}$. Assume that $\phi_n(x) = x^n +$ lower powers.

   (a) Show that $\phi_n(-x) = (-1)^n \phi_n(x)$.

   (b) Show that ϕ_n is orthogonal to all polynomials of degree $\leq n-1$.

   (c) Show that $\phi_n(x)$ satisfies this recurrence relation:
   $$\phi_{n+1}(x) = x\phi_n(x) - c_n\phi_{n-1}(x),\ n \geq 1, \text{ where } c_n = \frac{\langle \phi_n, x^n \rangle}{\|\phi_{n-1}\|^2}.$$

3. Define $D[\phi] = \int_0^1 (\phi'^2 + q\phi^2)dx$ and $H[\phi] = \int_0^1 \phi^2 dx$. Throughout, we require that $\phi \in C^{(1)}[0,1]$ and that $\phi(0) = 0$.

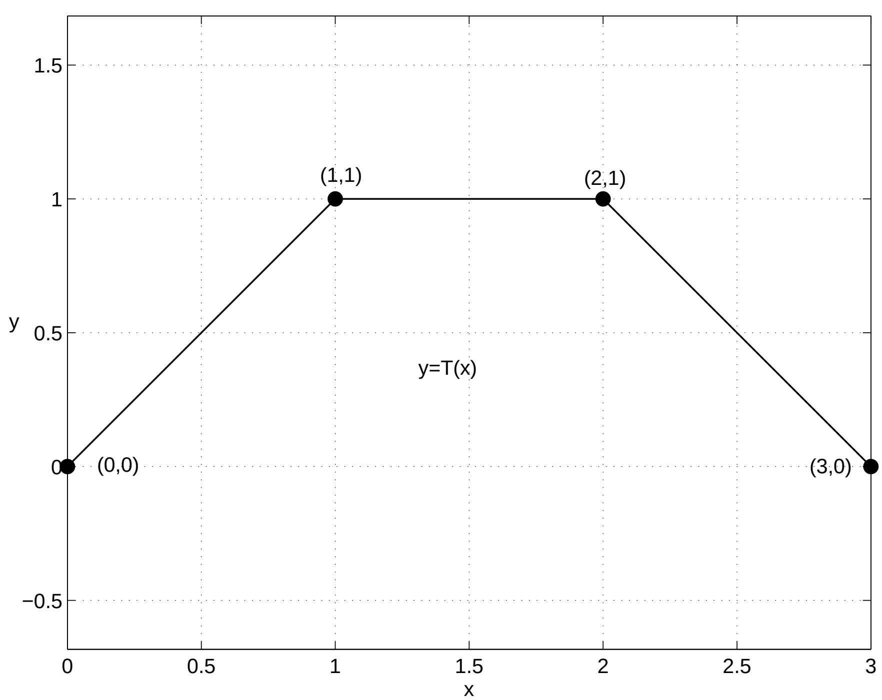
   (a) Let $\sigma \geq 0$. Minimize $D[\phi] + \sigma\phi^2(1)$ subject to the constraint $H[\phi] = 1$. Find the resulting Sturm-Liouville eigenvalue problem, including boundary conditions at $x = 1$.

(b) State the Courant Minimax Principle. Consider Dirichlet boundary conditons $\phi(0) = 0$, $\phi(1) = 0$. Order the first and second second eigenvalues for the two problems; that is if a, b, c, d are the four eigenvalues, then determine their aorder, $a \leq b \leq c \leq d$. Justify your answer.

4. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. The Fourier transform convention used here is $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{i\omega t}dt$.

   (a) Define convergence in $\mathcal{S}$. Sketch a proof: *The Fourier transform $\mathcal{F}$ is a continuous linear operator mapping $\mathcal{S}$ into itself.* Briefly explain how to use this to define the Fourier transform of a tempered distribution. This fails for $\mathcal{D}'$. Why?

   (b) You are **given** that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$. Let $T(x) = 0$ if $x \notin (0, 3)$. On $[0, 3]$, let T be the linear spline shown. Find $\widehat{T}$. (Hint: What is T''?)



Part 2: Numerical Analysis

Instructions: *Do all problems in this part of the exam. Show all of your work clearly.*

1. Consider the system

$$\begin{aligned} -\Delta u - \phi &= f \\ u - \Delta\phi &= g \end{aligned} \tag{1}$$

in the bounded, smooth domain Ω, with boundary conditions $u = \phi = 0$ on $\partial\Omega$.

(a) Derive a weak formulation of the system (1), using suitable test functions for each equation. Define a bilinear form $a\big((u,\phi),(v,\psi)\big)$ such that this weak formulation amounts to

$$a\big((u,\phi),(v,\psi)\big) = (f,v) + (g,\psi). \tag{2}$$

(b) Choose appropriate function spaces for u and ϕ in (2).

(c) Show, that the weak formulation (2) has a unique solution. Hint: Lax-Milgram.

(d) For a domain $\Omega_d = (-d,d)^2$, show that

$$\|u\|^2 \le cd^2\|\nabla u\|^2 \tag{3}$$

holds for any function $u \in H_0^1(\Omega_d)$.

(e) Now change the second "-" in the first equation of (1) to a "+". Use (3) to show stability for the modified equation on Ω_d, provided that d is sufficiently small.

2. Consider the two finite elements (τ, Q_1, Σ) and $(\tau, \widetilde{Q}_1, \Sigma)$, where $\tau = [-1,1]^2$ is the reference square and

$$\begin{aligned} Q_1 &= \operatorname{span}\big\{1, x, y, xy\big\}, \\ \widetilde{Q}_1 &= \operatorname{span}\big\{1, x, y, x^2 - y^2\big\}. \end{aligned}$$

$\Sigma = \{w(-1,0), w(1,0), w(0,-1), w(0,1)\}$ is the set of the values of a function $w(x,y)$ at the midpoints of the edges of τ.

(a) Which of the two elements is unisolvent? Prove it!

(b) Show that the unisolvent element leads to a finite element space, which is not H^1-conforming.

3. Consider the following initial boundary value problem: find $u(x,t)$ such that

$$
\begin{aligned}
u_t - u_{xx} + u &= 0, && 0 < x < 1,\ t > 0 \\
u_x(0,t) = u_x(1,t) &= 0, && t > 0 \\
u(x,0) &= g(x), && 0 < x < 1.
\end{aligned}
$$

(a) Derive the semi-discrete approximation of this problem using linear finite elements over a uniform partition of $(0,1)$. Write it as a system of linear ordinary differential equations for the coefficient vector.

(b) Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.

(c) Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0,1)$-norm.