Applied Analysis Part

January 10, 2023

Name: ____

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let A be an $n \times n$ self-adjoint matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State and prove the Courant-Fischer min-max theorem.

(b) Let $B = [b_1 \ b_2 \ b_3]$ be a real $n \times 3$ matrix, with b_1, b_2, b_3 being linearly independent. Assume that $\|x\| = 1$. If $q(x) = x^T A x$ and $\widehat{q}(x) = q(x)|_{B^T x = 0}$, show that

$$\lambda_1 \geq \max_{\|x\|=1} \widehat{q}(x) \geq \lambda_4.$$

Problem 2. Let $Lu = -(x^2 u')'$, $1 \leq x \leq 2$, with the domain of L given by

$$D_L := \{u \in L^2[1,2] : Lu \in L^2[1,2],\ u(1) = 0,\ u'(2) = 0\}.$$

The homogeneous solutions to $Lu = 0$ are x^{-1} and 1.

(a) Find the Green's function $g(x,y)$ for the problem $Lu = f$, $u \in D_L$.

(b) Show that $Ku := \int_0^1 g(\cdot, y) u(y) dy$ is a compact, self adjoint operator, and that 0 in not an eigenvalue of K.

(c) Without actually finding them, show that the eigenfunctions of L contain an orthonormal set that is complete in $L^2[0,1]$.

Problem 3. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Fredholm Alternative.

(b) State the Closed Range Theorem.

(c) Let $\mathcal{H} = L^2[0,1]$. Define the kernel $k(x,y) := x^3 y^2$ and let $Ku(x) = \int_0^1 k(x,y) u(y) dy$. Show that K is in $\mathcal{C}(\mathcal{H})$.

(d) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (c) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0,1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Sketch a proof of the following: *If f is a piecewise C^1, 2π-periodic function, and if $S_N = \sum_{n=-N}^{N} c_n e^{inx}$ is the N^{th} partial sum of the Fourier series for f, then, for every $x \in \mathbb{R}$,*

$$\lim_{N \to \infty} S_N(x) = \frac{f(x^+) + f(x^-)}{2}.$$

NUMERICAL ANALYSIS QUALIFIER

January, 2023

Problem 1. Let $\mathbb{P}_2$ be the space of polynomials in two variables spanned by $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, let $\hat{T}$ be the reference unit triangle, $\hat{\gamma}$ one side of $\hat{T}$, and $\hat{\pi}$ the standard Lagrange interpolant in $\hat{T}$ with values in $\mathbb{P}_2$.

Recall that there exists a constant C only depending on the geometry of $\hat{T}$ such that

$$\forall v \in H^3(\hat{T}), \inf_{p\in\mathbb{P}_2} \|v + p\|_{H^3(\hat{T})} \leq C|v|_{H^3(\hat{T})}.$$

1. State a trace theorem relating $L^2(\hat{\gamma})$ and $H^1(\hat{T})$.
2. Prove that there exists a constant $\hat{C}$ only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that

$$\forall \hat{u} \in H^3(\hat{T}), \|\hat{u} - \hat{\pi}(\hat{u})\|_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\hat{T})}.$$

3. Let Ω be a bounded polygon in $\mathbb{R}^2$, $\mathcal{T}_h$ be a triangulation of Ω and

$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_2\}.$$

   Let T be a triangle of $\mathcal{T}_h$ with diameter h_T and diameter of inscribed disc ϱ_T, and let γ be one side of T. Let F_T be the affine mapping from $\hat{T}$ onto T and let $\pi_{2,h}$ denote the standard Lagrange interpolant on X_h. Prove that there exists a constant C only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that

$$\forall u \in H^3(T), \|u - \pi_{2,h}(u)\|_{L^2(\gamma)} \leq C\sigma_T h_T^{2+1/2}|u|_{H^3(T)},$$

   where $\sigma_T = h_T/\varrho_T$.

Problem 2. For $f \in L^2(0,\ell)$, with $\ell \leq 1$, consider the following weak formulation: Seek $(u, v) \in \mathbb{V} := H_0^1(0,\ell) \times H_0^1(0,\ell)$ satisfying for all $(\phi, \psi) \in \mathbb{V}$

$$a((u,v);(\phi,\psi)) := \int_0^\ell u'\phi' + \int_0^\ell v'\psi' - \int_0^\ell v\phi = \int_0^\ell f\psi =: L(\psi). \tag{2.1}$$

1. What is the corresponding strong form satisfied by u (eliminate v)?
2. Show that for all $w \in H_0^1(0,\ell)$

$$\left(\int_0^\ell w^2\right)^{1/2} \leq \left(\int_0^\ell |w'|^2\right)^{1/2}.$$

3. Show that $a(\cdot;\cdot)$ coerces the natural norm on $\mathbb{V}$:

$$|||\phi,\psi||| := \left(\|\phi\|^2_{H^1(0,\ell)} + \|\psi\|^2_{H^1(0,\ell)}\right)^{1/2}$$

   and explicitly find a coercivity constant.
4. Let $\mathbb{V}_h$ be a finite dimensional subspace of $\mathbb{V}$. Show that there is a unique $(u_h, v_h) \in \mathbb{V}_h$ satisfying for all $(\phi_h, \psi_h) \in \mathbb{V}_h$

$$a((u_h,v_h);(\phi_h,\psi_h)) = L(\psi_h).$$

5. Prove the estimate

$$|||u - u_h, v - v_h||| \leq C_1 \inf_{(\phi_h,\psi_h)\in\mathbb{V}_h} |||u - \phi_h, v - \phi_h|||$$

   where C_1 is a constant independent of h (find C_1 explicitly).

6. You may assume that $u, v \in H_0^1(0,\ell) \cap H^2(0,\ell)$. Propose a discrete space $\mathbb{V}_h$ such that
$$|||u - u_h, v - v_h||| \leq C_2 h(\|u\|_{H^2(0,\ell)} + \|v\|_{H^2(0,\ell)})$$
for a constant C_2 independent of h. Justify your suggestion (you can assume the standard interpolation estimates hold).

Problem 3. Let b be a strictly positive constant and consider the problem: find $u(x,t)$ such that
$$\begin{aligned} &\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1,\ 0 < t \\ &u(x,0) = u_0(x), \quad 0 < x < 1, \\ &u(0,t) = u(1,t),\ t > 0 \end{aligned}$$
where u_0 is a smooth periodic function. Let J and N be positive integers, $x_i = ih$ for $i = 0, ..., J$ where $h = 1/J$ and $t_n = n\tau$ for $n \geq 0$ where $\tau = 1/N$. Also denote by u_j^n the approximation of $u(x_j, t_n)$.

Set $u_j^0 = u_0(x_j)$ and define reccursively u_j^n by the following *Lax* scheme
$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\tau b}{2h}(u_{j+1}^n - u_{j-1}^n), \quad j = 0, ..., J,$$
with the convention that $u_{-1}^n = u_{J-1}^n$ and $u_{J+1}^n = u_1^n$. Show that for all $j = 0, ..., J$ and $n \geq 0$
$$\min_i(u_i^0) \leq u_j^n \leq \max_i(u_i^0)$$
provided $\frac{\tau b}{h} \leq 1$.