**Complex analysis qualifying exam, August 2009.**

1. Give the statements of the following theorems:

(a) Jensen's formula;
(b) Harnack's principle.

2. A continuous function on the real line is called analytic if it can be continuously and analytically extended to the upper half-plane. Similarly, a continuous function on the line is called antianalytic if it can be continuously and analytically extended to the lower half-plane. Prove that $f$ is analytic if and only if $\bar{f}$ is antianalytic.

3. Give an example of a real-valued harmonic function in the unit disk $\mathbb{D}$ that cannot be represented as a difference of two non-negative harmonic functions in $\mathbb{D}$.

4. Prove that the equation

$$az^3 - z + b = e^{-z}(z+2),$$

where $a > 0$ and $b > 2$, has two roots in the right half-plane $\Re z \geq 0$.

5. Let $f$ be an analytic function in the disk $\{|z| < R\}$ satisfying $|f(z)| < M$. Suppose that $f(z_0) = 0$ for some $z_0, |z_0| < R$. Show that then

$$|f(z)| \leq \frac{MR|z - z_0|}{|R^2 - z\bar{z}_0|}$$

for any $z, |z| < R$, and

$$|f'(z_0)| \leq \frac{MR}{R^2 - |z_0|^2}.$$

6. Let $f$ be an entire function. Let $a, b \in \mathbb{R}, a < b$. Prove that the residue of the function

$$f(z) \log \frac{z-b}{z-a}$$

at infinity is equal to

$$\int_a^b f(x)dx.$$

Here $\log \frac{z-b}{z-a}$ denotes any branch analytic in a neighborhood of infinity.

7. Let $f(z)$ be a function analytic in a domain containing the segment $[0, 1]$ and satisfying

$$f(z+1) = azf(z) + p(z)$$

in that domain, where $a \in \mathbb{R}$ and $p$ is a polynomial. Show that $f$ can be analytically continued to a domain $\{|\Im z| < \varepsilon\}$ for some $\varepsilon > 0$.

8. Find the function $w(z)$ that maps the domain $\Omega = \{\Im z > 0\} \setminus [0, i]$ conformally onto the unit disk and satisfies

$$w\left(\frac{5i}{4}\right) = 0, \quad w(i) = -i.$$

9. Let $f$ be analytic in $\mathbb{D} \setminus \{0\}$. Suppose that 0 is an essential singularity of $f$. Denote

$$M(r) = \max_{|z|=r} |f(z)|.$$

Prove that for any $p > 0$

$$\lim_{r \to 0+} r^p M(r) = \infty.$$

10. Find a general formula for an entire function of finite order that has no zeros.