Complex Analysis Qualifying Exam, August 2017.

Problem 1: Construct a conformal map from the unit disk onto the infinite horizontal strip $|\text{Im}(z)| < 1$.

Problem 2: Let f be a holomorphic function on an open, connected subset U of $\mathbb{C}$. Prove the following are equivalent:

(a) f is identically zero on U;
(b) There exists a point $a \in U$ such that $f^{(n)}(a) = 0$ for all $n \geq 0$;
(c) $Z(f) = \{ z \in U \mid f(z) = 0 \}$ has a limit point in U.

Problem 3: Suppose f is a continuous function on $\mathbb{C}$ which is holomorphic on $\mathbb{C} - \mathbb{R}$. Is f entire? Prove or give a counterexample.

Problem 4: Show the group of biholomorphic maps $f : \mathbb{C} \to \mathbb{C}$ consists of maps $z \mapsto az + b$ with $a \in \mathbb{C} - \{0\}$ and $b \in \mathbb{C}$.

Problem 5: Evaluate

$$\int_0^\infty \frac{\log(x)}{4 + x^2}\, dx$$

Problem 6: State and prove Hurwitz's theorem.

Problem 7: Let f be a polynomial of degree greater than 1. Let $K(f) = \{z \in \mathbb{C} \mid f^{[n]}(z) \nrightarrow \infty\}$ where $f^{[n]}$ is the n'th iterate of f. Then, $K(f)$ is a compact subset of $\mathbb{C}$. Let $J(f)$ be the boundary of $K(f)$. Show that:

$$J(f) = \{z \in \mathbb{C} \mid \text{the family } \{f^{[k]}\} \text{ is not normal on a neighborhood of } z\}.$$

Note: The constant function ∞ is an allowed locally uniform limit of a normal family.

Problem 8: Let K be a compact subset of $\mathbb{C}$ and G be a region which contains K. Let $\mathcal{P}$ be the set of polynomial functions on $\mathbb{C}$ and $\mathcal{O}(G)$ be the set of holomorphic functions on G. If $f : K \to \mathbb{C}$ is bounded let $||f||_K = \sup_{w \in K} |f(w)|$. Then, the polynomially convex hull of K and the holomorphically convex hull of K relative to G are defined to be the sets:

$$\hat{K} = \{z \in \mathbb{C} \mid |p(z)| \leq ||p||_K \ \ \forall p \in \mathcal{P}\}$$
$$\hat{K}_G = \{z \in G \mid |f(z)| \leq ||f||_K \ \ \forall f \in \mathcal{O}(G)\}$$

(a) Show that if $\mathbb{P}^1 - G$ is connected then $\hat{K}_G = \hat{K}$, where $\mathbb{P}^1$ is the Riemann sphere.
(b) Let K be the unit circle. Determine $\hat{K}_G$ in the cases $G = \mathbb{C}$ and $G = \mathbb{C} - \{0\}$.

Problem 9: Give an explicit construction of an entire function f which has a simple zero at $m + in$ for all $m, n \in \mathbb{Z}$.

Problem 10: State and prove a Schwarz reflection principle for harmonic functions.