Complex Analysis Qualifying Exam, January 2020

Problem 1: Let $S = \{z \in \mathbb{C} \mid e^{e^z} = 1\}$. Find the distance from S to the point i, that is, find $\inf_{z \in S} |z - i|$.

Problem 2: (a) Show that there exists an analytic function f in the open right half-plane such that $(f(z))^2 + 2f(z) \equiv z^2$.

(b) Show that your function f can be continued analytically to a region containing the set $\{z \in \mathbb{C} \mid |z| = 3\}$.

Problem 3: Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of complex numbers for which $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} d_n = 1$. Prove that for n sufficiently large, the polynomials $p_n(z) = a_n + b_n z + c_n z^2 + d_n z^3$ have three distinct zeros.

Problem 4: Find all entire functions f which satisfy $f(0) = 0$, $f'(0) = 1$ and the family of successive iterates $\{f, f \circ f, f \circ f \circ f, \ldots\}$ is normal. (The $\circ$ denotes composition.)

Problem 5: f_1 and f_2 are analytic in the unit disc. Show that if $|f_1|^2 + |f_2|^2 \equiv 1$, then f_1 and f_2 are constant.

Problem 6: (a) State (but do not prove) the Riemann mapping theorem.

(b) Let $a \in \mathbb{C}$ and $\Omega = \mathbb{C} \setminus \{a\}$. Can one map Ω analytically onto the unit disc? Justify your answer. If your answer is 'yes', find such a map.

(c) Let $\Omega = \mathbb{C} \setminus [0, \infty)$. Can one map Ω conformally (i.e. analytically, one-to-one) onto the unit disc? Justify your answer. If your answer is 'yes', find such a map.

Problem 7: (a) For $a \in \mathbb{C}$ with $0 < |a| < 1$ and $|z| \le r < 1$, show that

$$\left| \frac{a + |a|z}{(1 - \overline{a}z)a} \right| \le \frac{1 + r}{1 - r}\,.$$

(b) Let $\{a_n\}$ be a sequence of complex numbers with $0 < |a_n| < 1$ and $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. Show that

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1 - \overline{a_n} z} \right)$$

converges locally uniformly in the unit disc, and that $|B(z)| \leq 1$.

(c) What are the zeros of $B(z)$? Justify your answer.

Problem 8: Compute

$$\int_0^\infty \frac{x^{1/3}}{(x+8)(x+1)^2}\, dx\ .$$

Problem 9: Suppose f is analytic in the strip $\{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1\}$ and continuous on its closure. Show that if $f(it) = f(1+it)\ \forall t \in \mathbb{R}$, then there is an entire function g such that $f(z) = g(z),\ 0 \leq \mathrm{Re}(z) \leq 1$.

Problem 10: Show that there exists a sequence $\{f_n\}$ of entire functions with the following property: for each rational number $q \geq 0$, the sequence converges to $\sqrt{q}$ uniformly on compact subsets of the line $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = q\}$.