Qualifying Examination in Real Variables, August 2015

General Instructions:

(1) For each problem, use a new sheet.

(2) All vector spaces are over $\mathbb{R}$ and all functions are $\mathbb{R}$-valued.

(3) Unless otherwise stated, you may use results from Folland's book, but you need to state them carefully (it is not necessary to remember their names).

Problems:

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. For each $t \in \mathbb{R}$ define
$$f_t(x) = f(t + x), \quad x \in \mathbb{R}.$$
Prove that $f_t(x)$ is a Borel measurable function (in x) for each fixed $t \in \mathbb{R}$.

(2) Justify the statement that
$$\int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2}\, dx\, dy = \int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2}\, dy\, dx.$$

(3) Assume that (f_n) is a sequence in $C[0,1]$.
   a) Show that (f_n) converges weakly to 0 if and only if (f_n) is bounded in $C[0,1]$ and $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in [0,1]$.
   b) Show that if (f_n) converges weakly in $C[0,1]$, then it converges in norm in $L_p[0,1]$ for all $1 \le p < \infty$.

(4) Let A be a Lebesgue null set in $\mathbb{R}$. Prove that
$$B := \{e^x : x \in A\}$$
is also a null set.

(5) a) Define absolute continuity of a function $f : \mathbb{R} \to \mathbb{R}$ and of a function $f : [a,b] \to \mathbb{R}$.
   b) Show that if f and g are absolutely continuous on $[a,b]$, $a, b \in \mathbb{R}$, $a < b$, then $f \cdot g$ is absolutely continuous on $[a,b]$.
   c) Give an example to show that (b) is false if $[a,b]$ is replaced by $\mathbb{R}$.

(6) Let X and Y be Banach spaces and $T : X \to Y$ be a one-to-one, bounded and linear operator for which the range $T(X)$ is closed in Y. Show that for each continuous linear functional ϕ on X there is a continuous linear functional ψ on Y, so that $\phi = \psi \circ T$.

(7) State the Open Mapping Theorem and the Closed Graph Theorem for Banach spaces. Derive the Open Mapping Theorem from the Closed Graph Theorem.

(8) Let Y be a closed subspace of a Banach space X, with norm $\|\cdot\|$. Let $|||\cdot|||$ be a norm on Y which is equivalent to $\|\cdot\|$, meaning that there is a $C \geq 1$ so that

$$\frac{1}{C}|||y||| \leq \|y\| \leq C|||y||| \text{ for all } y \in Y.$$

Let S be the set of all linear functionals $\phi : X \to \mathbb{R}$, so that

(a) $$|\phi(y)| \leq |||y||| \text{ for all } y \in Y, \text{ and}$$

(b) $$|\phi(x)| \leq C\|x\| \text{ for all } x \in X.$$

Prove the following statements

i) $|||x||| := \sup_{\phi \in S} |\phi(x)|$, $x \in X$, defines a norm on X.

ii) $|||y||| = |||y|||$ for $y \in Y$.

iii) The norms $|||\cdot|||$ and $\|\cdot\|$ are equivalent on X.

(9) Let f be increasing on $[0,1]$ and let

$$g(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \qquad \text{for } 0 < x < 1.$$

Prove that if $A = \{x \in (0,1) : g(x) > 1\}$ then

$$f(1) - f(0) \geq m^*(A).$$

(10) a) State a version of the Stone-Weierstrass Theorem.

b) Let A be a uniformly dense subspace of $C[0,1]$ and let

$$B = \Big\{F(x) : F(x) = \int_0^x f(t)\,dt, \quad 0 \leq x \leq 1, f \in A\Big\}.$$

Prove that B is uniformly dense in

$$C_0[0,1] := \{g \in C[0,1] : g(0) = 0\}.$$

c) Prove that the span of $\{\sin(nx) : n \in \mathbb{N}\}$ is dense in $C_0[0,1]$.