# Real Analysis Qualifying Exam
August, 2021

**1.** Let $(X, \Omega)$ be a measurable space and suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of real valued measurable functions on $X$. Show that the set all of points $x \in X$ for which $(f_n(x))_{n=1}^{\infty}$ converges is a measurable set.

**2.** Suppose that $X$ is a linear subspace of $L^{2021}([0,1])$ that is closed as a subspace of $L^1([0,1])$. Show that $X$ is closed as a subspace of $L^{2021}([0,1])$ and that $(X, \|\cdot\|_{2021})$ is isomorphic (meaning "linearly homeomorphic") to a Hilbert space.

**3.** Regard $L^\infty(0,1) = L^1(0,1)^*$. Prove that if $f \in L^\infty(0,1)$, then there is a sequence $(p_n)_{n=1}^{\infty}$ of polynomials such that $(1_{(0,1)}p_n))_{n=1}^{\infty}$ converges weak* to $f$.

**4.** Let $a$ and $b$ be real numbers satisfying $a > b > 1$. Evaluate

$$\lim_{n\to\infty} \int_0^\infty \frac{n\,|\cos(x)|}{1+n^a x^b}\,dx.$$

**5.** (a) (3 pts) State the closed graph theorem.
(b) (7 pts) Let $\alpha_n > 0$ (for $n \in \mathbb{N}$). Suppose that for any numbers $\gamma_n \geq 0$, we have

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \frac{\gamma_n}{\sqrt{\alpha_n}} < \infty.$$

Show that we must have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n} < \infty.$$

**6.** Prove that if $X$ is a separable Banach space then there is an injective bounded linear operator from $X$ into $\ell^{2021}$.

**7.** Prove that if $\mathcal{C}$ is a **weakly** compact subset of $C[0,1]$, then $\mathcal{C}$ is a **norm** compact subset of $L^2(0,1)$. You may use the theorem that if a subset of a Banach space is weakly compact then it is weakly sequentially compact.

**8.** (a) (5 pts) Prove that every infinite dimensional vector space contains a linearly independent set whose linear span is the whole space.
(b) (5 pts) Prove that every infinite dimensional Banach space has a discontinuous linear functional.

**9.** Prove or disprove: There is a continuous function $f$ from the reals to the reals such that for all rational numbers $x$, $f(x)$ is irrational, and for all irrational numbers $x$, $f(x)$ is rational.

**10.** Let $F$ be a continuous, real-valued function on $[0,1] \times [0,1] \times [-1,1]$ and for $f$ in the unit ball of $C_{\mathbb{R}}[0,1]$, define $G_f : [0,1] \to \mathbb{R}$ by

$$G_f(s) = \int_0^1 F(s,t,f(t))\,dt.$$

Show that $\{G_f \mid f \in C_{\mathbb{R}}[0,1],\ \|f\| \leq 1\}$ is a pre-compact subset of $C_{\mathbb{R}}[0,1]$.