Real Analysis Qualifying Exam
January 2011

Each problem is worth ten points. Work each problem on a separate piece of paper.

1. Working directly from the definition of almost everywhere convergence, prove that if $\{f_n\}_{n=0}^{\infty}$ is a sequence of measurable functions on a measure space $(X, \mathscr{M}, \mu)$ such that $\int_X |f_n - f_0|^{1/4}\, d\mu < n^{-2}$ for each n then $\{f_n\}_{n=1}^{\infty}$ converges to f_0 μ-almost everywhere.

2. Let K be a compact metric space. Show that $C(K)$ is separable.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a finite measure space $(X, \mathscr{M}, \mu)$. Recall that $\{f_n\}_{n=1}^{\infty}$ is said to be *uniformly integrable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_E f_n\, d\mu| < \varepsilon$ for all measurable sets $E \subseteq X$ satisfying $\mu(E) < \delta$ and all n. Prove that if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable, $\sup_n \|f_n\|_1 < \infty$, and $\{f_n\}_{n=1}^{\infty}$ converges in measure to 0, then $\|f_n\|_1 \to 0$ as $n \to \infty$.

4. Let $1 \le p < \infty$ and let f be a positive element of $L^p[0,1]$. Prove that the set $\{f^{1/n} : n \in \mathbb{N}\}$ has compact closure in $L^p[0,1]$. Give an example to show that this is false when $p = \infty$.

5. Let X be a reflexive Banach space and K a nonempty closed convex subset of X. Prove that there exists an $x \in K$ such that $\|x\| = \inf_{y \in K} \|y\|$. Show that this x is unique in the case that X is a Hilbert space.

6. Let X be a Banach space such that X^* is separable. Prove that X is separable.

7. (a) State what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *absolutely continuous*.

   (b) Let $F : \mathbb{R} \to \mathbb{R}$ be a function and let $0 \le M < \infty$. Show that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'(x)| \le M$ almost everywhere with respect to Lebesgue measure.

8. For a function $f : [0,1] \to \mathbb{R}$ define

$$\|f\|_L = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : 0 \le x < y \le 1\right\}.$$

   Prove that the set of all functions $f : [0,1] \to \mathbb{R}$ satisfying $\|f\|_L < \infty$ is dense in $L^1[0,1]$.

9. Let $g : [0,1] \to [0,1]$ be a continuous function. Determine, with proof, conditions on g which are equivalent to the property that $\lim_{n\to\infty} \|g^n f\|_2 = 0$ for all $f \in L^2[0,1]$.

10. (a) State Fubini's theorem.

    (b) Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $C([0,1]^2)$. Suppose that $f_{x,n} \to 0$ weakly in $L^2(\mu)$ for every $x \in [0,1]$, where $f_{x,n}(y) = f_n(x,y)$ for all $y \in [0,1]$ and μ is Lebesgue measure on $[0,1]$. Prove that $f_n \to 0$ weakly in $L^2(\mu \times \mu)$.