Qualifying Examination in Real Variables, January 2014

General Instructions:

(1) Use for each problem an extra sheet.

(2) Unless stated otherwise, you may use results from Folland's book, but you need to state them carefully (it is not necessary to remember their names).

Problems:

(1) Let $(X, \mathcal{M}, \mu)$ be a non atomic measure space with $\mu(X) > 0$. Show that there is a measurable $f : X \to [0, \infty)$, for which

$$\int f(x)\, d\mu(x) = \infty.$$

(2) Assume that μ is a finite measure on $\mathbb{R}^n$. Prove that there is a closed set $A \subset \mathbb{R}^n$ with the property that for each closed $B \subsetneq A$ it follows that $\mu(A \setminus B) \neq 0$.

(3) For a nonnegative function $f \in L_1([0,1])$, prove that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dx = m\big(\{x : f(x) > 0\}\big).$$

(4) Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t) dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x) dx = \int_0^1 f(x) dx.$$

(5) Assume that ν and μ are two finite measures on a measurable space $(X, \mathcal{M})$. Prove that

$$\nu \ll \mu \iff \lim_{n\to\infty} (\nu - n\mu)^+ = 0.$$

(6) Let (p_n) be a sequence of polynomials which converges uniformly on $[0,1]$ to some function f, and assume that f is not a polynomial. Prove the $\lim_{n\to\infty} \deg(p_n) = \infty$, where $\deg(p)$ denotes the degree of a polynomial p.

(7) Let (f_n) be sequence of non zero bounded linear functionals on a Banach space X. Show that there is an $x \in X$ so that $f_n(x) \neq 0$, for all $n \in \mathbb{N}$.

(8) Assume that $T : \ell_1 \to \ell_2$ is bounded, linear and one-to-one. Prove that $T(\ell_1)$ is not closed in ℓ_2.

(9) For a uniformly bounded sequence (f_n) in $C[0,1]$ (i.e. $\sup_{n\in\mathbb{N}} \sup_{\xi\in[0,1]} |f_n(\xi)| < \infty$) show that

$$f_n \text{ converges weakly to } 0 \iff \lim_{n\to\infty} f_n(\xi) = 0 \text{ for all } \xi \in [0,1].$$

Is the equivalence true if we do not assume that (f_n) is uniformly bounded, explain?

(10) Assume that f is a measurable and non negative function on $[0,1]^2$ and that $1 \le r < p < \infty$. Show that

$$\left(\int_0^1 \left(\int_0^1 f^r(x,y)\,dy\right)^{p/r} dx\right)^{1/p} \le \left(\int_0^1 \left(\int_0^1 f^p(x,y)\,dx\right)^{r/p} dy\right)^{1/r}.$$

Hint: Let $s = p/r$, let $1 < s' < \infty$ be the conjugate of s and let

$$F := [0,1] \to \mathbb{R}_0^+, \quad x \mapsto \int_0^1 f^r(x,y)\,dy.$$

Then consider for an appropriate function $h \in L_{s'}[0,1]$ the product hF.