TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2016

INSTRUCTIONS

- There are 9 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

1. Prove that a topological space X is discrete if and only if every map $f : X \longrightarrow Y$ for every topological space Y is continuous.
2. Prove that the closure of a connected set is connected.
3. Prove that if in a compact metric space the closure of any open ball is the closed ball of the same radius, then every ball of this space is connected.
4. Prove that the k-dimensional torus $\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{k \text{ times}}$ can be embedded into the $k+1$-dimensional space $\mathbb{R}^{k+1}$.
5. Prove that every bounded open convex nonempty set in the plane is homeomorphic to the plane.
6. In $\mathbb{R}^3$ let $\omega = xy\,dx + 2z\,dy - y\,dz$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the equation $f(u,v) = (uv, u^2, 3u+v)$. Calculate $d\,\omega$ and $f^*\omega$ and $f^*(d\omega)$ and $d(f^*\omega)$ directly.
7. (a) Let $F\colon M \to N$ be a smooth map between manifold M and N. Give the definitions of a regular point, a critical point and of a regular value and of a critical value of F.

   (b) Prove or disprove by giving a counter-example: If $c \in N$ is a critical value of F, then $F^{-1}(c)$ is not an embedded submanifold of M.

   (c) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$(x,y,z) \mapsto (x^3 + y^3 + z^3, z - xy).$$

   i) Find all $a \in \mathbb{R}$ such that $(a, 0)$ is a critical value of F;

   ii) For which $a \in \mathbb{R}$ is $F^{-1}(a, 0)$ an embedded submanifold of $\mathbb{R}^3$?
8. (a) Give the definition of an involutive distribution in terms of vector fields tangent to it.

(b) Let $X_1 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$, $X_2 = x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1}$ be two vector fields in $\mathbb{R}^3$.

i) Prove that for any point $p \in \mathbb{R}^3$ there are no neighborhood U and coordinate functions y_1, y_2, y_3 on U such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$.

ii) On the set $M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \neq 0\}$ define the distribution $D = \mathrm{span}(X_1, X_2)$. Prove that D is involutive and find the maximal connected integral submanifolds of it (in M).

(c) Let D be a distribution on a manifold M. Recall that a p-form ω annihilates D if

$$\omega(X_1, \dots, X_p) = 0$$

whenever $X_1, \dots, X_p$ are local sections of D. Prove that the distribution D is involutive (in the sense of definition given in item a)) if and only for any 1-form η that annihilates D the 2-form $d\eta$ annihilates D.

9. Let $M = \mathbb{R}^2$ with the standard coordinates (x, y). Consider the following Riemanian metric on M:

$$dx^2 + 2\cos\big(\alpha(x, y)\big)dxdy + dy^2,$$

where α is a smooth function on M such that $\alpha(x, y) \neq \pi k$, $k \in \mathbb{Z}$ for all $(x, y) \in \mathbb{R}$.

(a) Prove that the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\sin\big(\alpha(x, y)\big)}\Big(\frac{\partial}{\partial y} - \cos\big(\alpha(x, y)\big)\frac{\partial}{\partial x}\Big)$$

constitute an orthonormal frame with respect to this Riemannian metric.

(b) Find the dual coframe to the frame (e_1, e_2).

(c) Prove that the Gaussian curvature of this Riemannian metric is equal to $-\frac{\alpha_{xy}}{\sin\alpha}$.