Topology/Geometry Qualifying Exam

(January 2012)

Remark: *In all solutions you must write a justification for your answers. Please start each problem in a new page.*

Problem 1. Given $P \in \mathbb{R}^2$ let $B_\epsilon(P)$ and $S_\epsilon(P)$ denote, respectively, the open Euclidean ball and the circle of radius ϵ centered at P. Let $X = B_1(0) \cup S_1(0)$ denote the closed ball of radius 1 centered at the origin $0 \in \mathbb{R}^2$. Given $P \in S_1(0)$ and $0 < \epsilon < 1/2$, define

$$N_\epsilon(P) = \{P\} \cup B_\epsilon\left((1-\epsilon)P\right),$$

where $(1-\epsilon)P$ denotes the multiplication of the vector $P \in \mathbb{R}^2$ by the scalar $(1-\epsilon)$. Note that $B_\epsilon((1-\epsilon)P)$ is tangent to $S_1(0)$ at P and that $N_\epsilon(P) = \cup_{0<r<\epsilon} S_r((1-r)P)$. (See picture below.)

Define

$$\mathcal{B} \;=\; \{N_\epsilon(P) \mid P \in S_1(0),\ 0<\epsilon<1/2\} \;\cup\; \{B_\rho(Q) \mid Q \in B,\ 0 < \rho < 1 - \|Q\|\}$$

where $\|Q\|$ denotes the Euclidean norm, and observe that $\mathcal{B}$ forms a **basis for a topology on X.**

a. Show that this topology is 1*st countable* and *separable*, but neither 2*nd countable* nor *Lindelöf*.

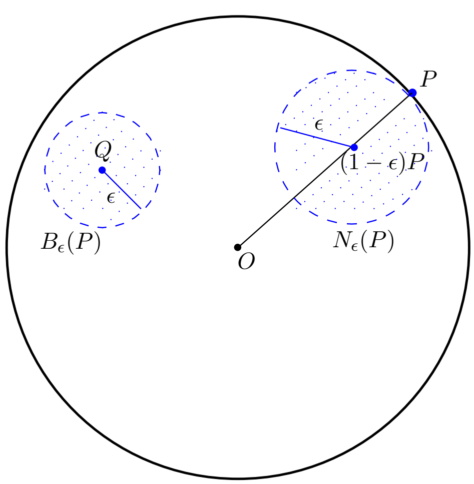
b. Describe the closure of $N_\epsilon(P)$ and prove that X is a *regular space*.

c. Fix $P \in S_1(0)$ and $0 < \epsilon < 1/2$. Define $f\colon X \to [0,1]$ by:

$$f(x) = \begin{cases} 1, & \text{if } x \in X - N_\epsilon(P) \\ 0, & \text{if } x = 0 \\ r/\epsilon, & \text{if } x \in S_r(P) - \{P\} \text{ and } 0 < r < \epsilon. \end{cases}$$

Show that f is continuous.

d. Prove that X is a completely regular space.



Problem 2. Let X denote the following subspace of the Euclidean plane $\mathbb{R}^2$:

$$X = \{(x, y) \mid xy = \frac{1}{n}, \text{ for some } n \in \mathbb{N}\} \ \cup \ \mathbb{Q} \times \{0\} \ \cup \ \{0\} \times \mathbb{Q}.$$

a. Is X locally-connected?

b. Describe the connected components and the quasicomponents of X.

Problem 3. Let X be a second countable, locally compact, Hausdorff space.

a. Show that the one-point compactification X_+ of X is second countable.

b. Show that every subspace $A \subset X$ is paracompact.

Problem 4. Let $f\colon X \to Y$ be a function from a space X into a locally compact, Hausdorff space Y. Show that f is continuous if and only if the following holds: *whenever $\hat{Y}$ is a compact Hausdorff space containing Y as a subspace, then the graph of f is a closed subspace of $X \times \hat{Y}$.*

Problem 5. Consider $\mathbb{R}^3$ with the Euclidean metric. Let $S \subset \mathbb{R}^3$ be a smooth oriented surface and $C \subset S$ a regular, oriented curve. Let $\mathrm{SO}(3)$ be the set of oriented orthonormal framings (or bases) of $\mathbb{R}^3$. Construct a smooth moving frame $e = (e_1, e_2, e_3) : C \to \mathrm{SO}(3)$ as follows: given $x \in C$, let $e_1(x) \in T_xC$ be the oriented unit tangent vector to C; let $e_3(x) \in N_xS$ be the oriented unit normal vector to S; set $e_2(x) = e_3(x) \times e_1(x) \in T_xS$.

a. Show that there exist 1-forms $\alpha, \beta, \gamma \in \Omega^1(C)$ on C such that

$$\mathrm{d}e_1 = \alpha e_2 + \beta e_3 , \quad \mathrm{d}e_2 = -\alpha e_1 + \gamma e_3 \quad \text{and} \quad \mathrm{d}e_3 = -\beta e_1 - \gamma e_2 .$$

b. Show that C is a geodesic if and only if $\alpha = 0$.

c. Compute the curvature κ of C in terms of α, β, γ.

d. Let II denote the second fundamental form of S. Compute $\mathrm{II}(e_1, e_1)$ in terms of α, β, γ.

Problem 6. Let M be a smooth n–dimensional manifold. Let $X_1, \ldots, X_k$ be point-wise linearly independent vector fields on M, and let $\omega^1, \ldots, \omega^{n-k}$ be point-wise linearly independent 1-forms such that

$$\mathrm{Ann}\{\omega^1, \ldots, \omega^{n-k}\} \ = \ \mathrm{span}_{\mathcal{C}^\infty(M)}\{X_1, \ldots, X_k\} ,$$

where Ann denotes the annihilator in the dual space. Prove that $[X_a, X_b] \equiv 0$ modulo $X_1, \ldots, X_k$, for all $1 \leq a, b \leq k$, if and only if $\mathrm{d}\omega^s \equiv 0$ modulo $\omega^1, \ldots, \omega^{n-k}$ for all $1 \leq s \leq n-k$.

Problem 7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = (x-1)^2 - yz$. For which $t \in \mathbb{R}$ is $f^{-1}(t)$ an embedded 2-dimensional submanifold of $\mathbb{R}^3$?

Problem 8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $f(x, y) = 0$ for all (x, y) outside the unit disk, i.e., for all (x, y) with $x^2 + y^2 \geq 1$. Consider the surface in $\mathbb{R}^3$ given by the graph of f. What can you say about the average Gauss curvature of the surface?

Problem 9. Consider the tangent bundle to the 3-sphere. Is it a smooth manifold? If so, what is its dimension? Is it compact?

Problem 10. Either prove the projective plane $\mathbb{RP}^2$ is not orientable, or find an orientation for it. (Hint: construct $\mathbb{RP}^2$ as the 2-sphere quotiented by the antipodal map.)