TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2015

INSTRUCTIONS

- `There are 8 problems. Work on all of them.`
- `Prove your assertions.`
- `Use a separate sheet of paper for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

1. Let C be a subset of a topological space X.
   (a) Prove that if C is connected, then the closure of C is connected.
   (b) Prove or give a counter-example to the following statement: if C is connected, then the interior of C is connected.

2. Prove that a separable metric space (a metric space with a dense countable subset) is second countable.

3. Let $\mathbb{R}_\ell$ denote the real line $\mathbb{R}$ with the lower limit topology i.e. the topology on $\mathbb{R}$ with the basis consisting of all left-closed, right-open intervals $[a, b)$.
   (a) Find the closure of the set (a, b) in $\mathbb{R}_\ell$;
   (b) Prove that $\mathbb{R}_\ell$ is not locally compact space.

4. (a) Give the definition of a quotient map $q : X \to Y$ between two topological spaces X and Y;
   (b) Let $q : X \to Y$ be an open quotient map. Show that the space Y is Hausdorff if and only if the set $A = \{(x_1, x_2) \in X \times X |\ q(x_1) = q(x_2)\}$ is closed in the product space $X \times X$.

5. Let X be a copy of real line $\mathbb{R}$ and let $\phi : X \to \mathbb{R}$ be $\phi(x) = x^5$. Taking ϕ as a chart, this defines a smooth structure on X. Prove or disprove each of the following statements:
   (a) X (with this smooth structure) is diffeomorphic to $\mathbb{R}$.
   (b) ϕ together with the identity map comprise a smooth atlas.

6. (a) Give the definition of an involutive smooth distribution in terms of its smooth local sections.

   (b) One says that a $p-$form ω annihilates a smooth distribution D if $\omega(X_1, \ldots, X_p) = 0$ whenever $X_1, \ldots X_p$ are local sections of D. Based on the definition of the previous item prove that a smooth distribution D is involutive if an only if the following condition is satisfied: if η is any smooth 1-form that annihilates D on an open subset $U \subset M$, then $d\eta$ annihilated D on U.

   (c) Given vector fields $X_1 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial w}$ in $\mathbb{R}^4$ with coordinates (x, y, z, w), can we find a two dimensional submanifold M of $\mathbb{R}^4$ such that X_1 and X_2 are tangent to M at every point of M? Prove your answer.

   (d) With the same notations as in the previous item, can we find a two dimensional submanifold N such that X_1 is tangent to N at every point of N? Prove your answer.

7. (a) Give a definition of an embedded submanifold of a manifold N.

   (b) Let $F : M \mapsto N$ be a smooth map between manifolds M and N. Give the definitions of a regular point and of a regular value of F.

   (c) Recall that a matrix A with real entries is orthogonal if $A^T A = I$, where I is the identity matrix. Prove that the set $O(n)$ of all $n \times n$ orthogonal matrices is an embedded submanifold of the space $M(n, \mathbb{R})$ of all $n \times n$ matrices with real entries and find the dimension of this submanifold.

8. (a) Let $\mathbb{R}^2$ have coordinates (u, v) and fix a constant $a > 0$. The *catenoid* is the image of the mapping $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by $f(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$. Compute the mean curvature and the Gaussian curvature of the catenoid.

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $f(x, y) = 0$ for all (x, y) outside the unit disk, i.e., for all (x, y) with $x^2 + y^2 \geq 1$. Consider the surface S in $\mathbb{R}^3$ given by the graph of f over the disk $x^2 + y^2 \leq 2$. What can you say about the integral of the Gaussian curvature over S? Prove your answer.