TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
JANUARY 2016

INSTRUCTIONS

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Problem 1. Show that a bijection $f : X \longrightarrow Y$ is a homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$.

Problem 2. Prove that the one-point compactification of the half-open interval $[0, 1)$ is homeomorphic to the closed interval $[0, 1]$.

Problem 3. a) Give the definition of a connected component of a topological space.

b) Let X be a topological space, and let $X' \subset X$. Show that the connected component of $x \in X'$ in the subspace X' is a subset of the connected component of x in X.

Problem 4. Prove that a metric space X is compact if and only if every continuous function $f : X \longrightarrow \mathbb{R}$ is bounded.

Problem 5. Let $\mathbb{R}^3$ have coordinates (x, y, z) and the standard Euclidean structure. Let $S \subset \mathbb{R}^3$ be the surface parametrized locally by $x = t + s$, $y = t^2 + 2ts$, $z = t^3 + 3st^2$, where $s, t > 0$. Using any method you please, determine the Gauss curvature function $K(s, t)$.

Problem 6. Let M be a differentiable manifold, and let $x \in M$.

(1) Define (without reference to the tangent space), the cotangent space of M at x, T_x^*M.
(2) Define (without reference to the cotangent space), the tangent space of M at x, T_xM.
(3) Show that, with the above definitions, T_xM and T_x^*M are dual vector spaces.

Problem 7. Consider the following Lie subgroup of $\mathrm{GL}_3\mathbb{R}$:

$$G := \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

Determine the Lie algebra of G as a subalgebra of $\mathfrak{gl}_3\mathbb{R}$.

Problem 8. On $\mathbb{R}^3$ with coordinates (x, y, z), let $\theta = dx + f(z)dy$, for some function $f(z)$. State a necessary and sufficient condition on $f(z)$ such that for each point of $\mathbb{R}^3$ there exists a surface S whose tangent plane is annihilated by θ, i.e, if v, w is a basis for T_pS, then $\theta_p(v) = 0$ and $\theta_p(w) = 0$.