TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2017

INSTRUCTIONS

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.
- Please do not interpret any problem in a way that renders it trivial.

1. Consider the following two subspaces of $\mathbb{R}$ with its usual topology:

$$X = (0,1) \cup \{2\} \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \dots$$
$$Y = (0,1] \cup \{2\} \cup (3,4) \cup \{5\} \cup \cdots \cup (3n, 3n+1) \cup \{3n+2\} \cup \dots$$

(a) Provide two bijective continuous functions $f : X \to Y$ and $g : Y \to X$ (please provide a carefully written proof that the functions you defined are continuous).

(b) Prove that X and Y are not homeomorphic.

2. Let X be a topological space. A family $\mathcal{F}$ of subsets of X is called *locally finite* if every point $x \in X$ has an open neighborhood U such that only finitely many members of $\mathcal{F}$ have nonempty intersection with U.

Prove that if $\mathcal{F}$ is a locally finite family of closed sets then the union $\bigcup_{C \in \mathcal{F}} C$ is closed.

3. Prove that every compact Hausdorff space is regular.

4. Let X and Y be topological spaces, Y a connected space, and $f : X \to Y$ a continuous surjective function such that $f^{-1}(y)$ is connected for every $y \in Y$.

(a) Show that, if f is a quotient map, then X is connected.

(b) Provide an example in which f is not a quotient map and X is not connected.

5. (a) Give the definition of a smooth n-dimensional manifold ("smooth" here means C^∞) and the definition of a smooth map between two manifolds;

   (b) Formulate the Inverse Function Theorem (for a smooth map between manifolds of the same dimensions);

   (c) Assume that $F : M \to N$ is a smooth map, $\dim M \geq \dim N$, and $\operatorname{rank} dF(m) = \dim N$ for some point $m \in M$. Prove that there exists an open neighborhood V of the point $F(m)$ in N that belongs to the image of F.

6. Let I_n be the $n \times n$ identity matrix and J be the $2n \times 2n$ matrix defined by $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Let $\mathrm{Sp}_{2n}(\mathbb{R})$ be the subset of $2n \times 2n$ matrices Q with real entries such that $Q^T J Q = J$.

(a) Prove that the set $\mathrm{Sp}_{2n}(\mathbb{R})$ is an embedded submanifold of the space of all $2n \times 2n$-matrices with real entries and find the dimension of this submanifold.

(b) It is known that $\mathrm{Sp}_{2n}(\mathbb{R})$ is a Lie subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$. Determine the Lie algebra of $Sp_{2n}(\mathbb{R})$ as a subalgebra of $\mathfrak{gl}_{2n}(\mathrm{R})$, namely if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a $2n \times 2n$-matrix, where A, B, C, D are $n \times n$-matrices, give explicit relations on matrices A, B, C, D such that X belongs to the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{R})$.

7. (a) Given a smooth map $F : M \to N$ between manifolds M and N and a differential p-form ω on a manifold N define the pull-back $F^*\omega$ of ω by F;

(b) Let $\{\omega_{ij}\}_{1\leq i,j\leq k}$ be the set of k^2 differential 1-forms on a manifold M. Let $E = M \times \mathbb{R}^k$ and $(y_1, \dots y_k)$ be the standard coordinates in $\mathbb{R}^k$. Also let $\pi : E \to M$ be the projection onto the first factor. For every $1 \leq j \leq k$ define the following 1-form α_j on E:

$$\alpha_j = dy_j + \sum_{l=1}^{k} y_l \pi^* \omega_{jl}.$$

Consider the distribution H on E defined by the forms $\alpha_1, \dots \alpha_k$, i.e. such that

$$H(p) = \{X \in T_pE : \alpha_1|_p(X) = \ldots = \alpha_k|_p(X) = 0\}$$

for every $p \in E$. Prove that the distribution H is involutive if and only if

$$d\omega_{ij} + \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} = 0$$

for all $1 \leq i, j \leq k$. (In your solution you can use the description of involutivity of the distributions H in terms of the ideal of forms, annihilating H.)

8. Let M be a 2-dimensional Riemannian manifold. Let (X_1, X_2) be an orthonormal frame (with respect to the Riemannian metric) in an open set U of M and (θ^1, θ^2) be the corresponding dual coframe.

(a) Prove that there exists the unique 1-form ω on U such that

$$d\theta^1 = -\omega \wedge \theta^2, \quad d\theta^2 = \omega \wedge \theta^1. \tag{1}$$

(b) Take another orthonormal frame $(\widetilde{X}_1, \widetilde{X}_2)$ on U, defining the same orientation on U as the frame (X_1, X_2). Let $(\widetilde{\theta}^1, \widetilde{\theta}^2)$ be the corresponding dual coframe, and $\widetilde{\omega}$ be the 1-form satisfying relation (1) with (θ^1, θ^2) replaced by $(\widetilde{\theta}^1, \widetilde{\theta}^2)$. Prove that $d\widetilde{\omega} = d\omega$. What is the analogous formula between $d\omega$ and $d\widetilde{\omega}$ if $(\widetilde{X}_1, \widetilde{X}_2)$ and (X_1, X_2) define the opposite orientations on U.

(c) Define the Gaussian curvature of the 2-dimensional Riemannian manifold M based on the constructions of the previous items.