TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2023

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- There are 10 problems. Try to solve 4 problems out of Problems $1-5$ and 4 problems out of Problems 6—10 (you can try to solve all 10 problems as well, every extra problem will be graded as a bonus). Try to write as complete solutions and proofs as possible.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

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1. (a) Given two manifolds $M$ and $N$ and a smooth map $F : M \to N$ define regular points and regular values of $F$.

   (b) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$F(x, y, z) = (x^2 + 2y^2 + 3z^2, 3x + 4y + 3z).$$

   For what values of $t$ the set $F^{-1}(1, t)$ is an embedded submanifold of $\mathbb{R}^3$? For each such $t$ calculate the dimension of $F^{-1}(1, t)$.

2. (a) Formulate the Frobenius theorem.

   (b) Consider the 1- form $\alpha = dx + zdy + zdz$ on $\mathbb{R}^3$. Does an injective immersion $F : \mathbb{R}^2 \to \mathbb{R}^3$ exist such that $F^*\alpha = 0$? Prove your answer.

3. (a) Describe as detailed as possible how to define the integral of a differential $n$-form with compact support on an oriented $n$-dimensional manifold $M$.

   (b) Prove that if $\omega$ is a closed 2-form on $\mathbb{S}^4$ (the 4-dimensional sphere), then the 4-form $\omega \wedge \omega$ must vanish somewhere in $\mathbb{S}^4$ (*you can use here without proof that any closed* 2-*form on* $\mathbb{S}^4$ *is exact*).

4. (a) Given a vector field $X$ and a differential $p$-form $\omega$ on a manifold $M$ define the Lie derivative $L_X\omega$ of $\omega$ with respect to the vector field $X$, involving the flow generated by $X$.

   (b) Given a differential 1-form $\omega$ and two vetor fields $X$ and $Y$ prove that

$$(L_X\omega)(Y) = X\big(\omega(Y)\big) - \omega([X, Y]). \tag{1}$$

   (c) Let $M = \mathbb{R}^n$ with coordinates $(x^1, \dots, x^n)$. If $X = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$ for some smooth functions $v^i$, $1 \le i \le n$, express $L_X(dx^j)$ in terms of $v^i$s.

   (d) Write the analog of the formula (1) for differential $p$-forms (no need to prove this formula).

5. Let $\mathrm{Gr}_3(\mathbb{R}^4)$ be the set of all 3-dimensional subspaces in $\mathbb{R}^4$. Describe as completely as possible how to endow $\mathrm{Gr}_3(\mathbb{R}^4)$ with the structure of smooth manifold.

6. Recall that a topological space $X$ is *locally compact* if for any point $x$ there is a compact subspace $C$ of $X$ that contains an open neighborhood of $x$.

   (a) Show that the set of rational numbers $\mathbb{Q}$, considered as a subspace of the space of real numbers together with the standard topology $\mathbb{R}$, is not locally compact.

   (b) Let $X$ be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact as a subspace of $Y$?

7. Let $f : S^1 \to \mathbb{R}$ be a continuous map from the circle to the real numbers. Show that there exists a point $x$ of $S^1$ such that $f(x) = f(-x)$.

8. Let $X$ be a metric space with metric $d$.

   (a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

   (b) Show that the topology induced by $d$ is the coarsest topology with respect to which the function $d$ is continuous.

9. Compute the fundamental group of the closed orientable surface of genus 2.

10. Show that if $n > 1$, then every continuous map $f : S^n \to S^1$ is nullhomotopic.