QUALIFYING EXAM-REAL ANALYSIS
AUGUST 2023

The 10 problems below are equally weighted. Solve as many problems or portions thereof as you can in 4 hours. Please start the solution of each problem you attempt on a new sheet in your bluebook. Make sure to properly mention any named theorem that you will need in any of your solutions.

In the sequel, unless specified otherwise, $\mathbb{R}$ (or a subset of it) is always equipped with the Borel σ-algebra and the Lebesgue measure (denoted by λ).

Problem 1. *Let $f\colon (0,1) \to \mathbb{R}$ be a Lebesgue integrable function. For any $x \in (0,1)$ define*

$$g(x) = \int_x^1 \frac{f(t)}{t} d\lambda(t).$$

Prove that $g\colon (0,1) \to \mathbb{R}$ is Lebesgue integrable and that $\int_0^1 f d\lambda = \int_0^1 g d\lambda$.

Problem 2.

(1) *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions on X. Define what it means that $(f_n)_{n\geqslant 1}$ converges in measure to a measurable function g.*
(2) *Give an example of a sequence of measurable functions that converges pointwise but not in measure.*
(3) *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions on X that converges pointwise to g. Show that $(f_n)_{n\geqslant 1}$ converges in measure to g.*

Problem 3.

(1) *Let $\{X_i\}_{i\in I}$ be a collection of topological spaces. Define what is the product topology on the Cartesian product $\prod_{i\in I} X_i$.*
(2) *Prove that a compact metric space is separable.*
(3) *Show that every compact metric space is homeomorphic to a closed subset of $[0,1]^{\mathbb{N}}$ (equipped as usual with the product topology).*

Problem 4. *In this problem $\|f\|_\infty = \inf\{a \geqslant 0\colon \lambda(\{|f| > a\}) = 0\}$ is the essential norm of a measurable function. Let $L_\infty(\mathbb{R})$ denote the Banach space of functions that are essentially bounded and recall that $L_1(\mathbb{R})^* = L_\infty(\mathbb{R})$. Let $S = \{f \in L_\infty(\mathbb{R})\colon \lambda(\{x \in \mathbb{R}\colon |f(x)| > \frac{1}{1+e^{-|x|}}\}) = 0\}$.*

(1) *State the Banach-Alaoglu theorem.*
(2) *Show that S is a weak*-compact subset of $L_\infty(\mathbb{R})$.*
(3) *Is S a norm-compact subset of $L_\infty(\mathbb{R})$?*

Problem 5. *Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous and $C([0,1])$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$, endowed as usual with the sup-norm, i.e., $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$. For $f \in C([0,1])$ define $T(f)\colon [0,1] \to \mathbb{R}$ by:*

$$T(f)(x) = \int_0^1 k(x,y) f(y) dy, \ \ x \in [0,1].$$

(1) *Show that $T(C[0,1]) \subset C([0,1])$.*
(2) *Show that T maps bounded sets to subsets of compact sets.*

Problem 6.
Let $f\colon (0,\infty) \to \mathbb{R}$ *be continuous and such that for all* $x > 0$ *the sequence* $(f(nx))_{n=1}^{\infty}$ *converges to* 0*. Show that* $\lim_{x\to\infty} f(x) = 0$*.*

Problem 7.

(1) State the Hahn-Banach theorem (the version for sublinear and linear functionals on a real vector space).

For the next two questions, X *is a real normed vector space and we denote by* X^* *the Banach space of all bounded linear functionals on* X*.*

(2) Let C *be a convex open set in* X *that contains* 0 *and define for all* $x \in X$, $p_C(x) = \inf\{\alpha > 0\colon\ x \in \alpha C\}$*. Show that* p_C *is a sublinear functional on* X *and that* $C = \{x \in X\colon\ p_C(x) < 1\}$*.*

(3) Let C *be an open, convex, non-empty subset of* X *and* $x_0 \notin C$*. Show that there is* $x^* \in X^*$ *such that* $x^*(z) < x^*(x_0)$ *for all* $z \in C$*.*

Problem 8.

(1) Given a Banach space $(X, \|\cdot\|)$ *and a sequence* $(x_n)_{n\geqslant 1} \in X$*. What does it mean that* $(x_n)_{n\geqslant 1}$ *converges weakly to* $x \in X$*?*

(2) Let K *be a compact Hausdorff space and let* $C(K)$ *be the Banach space of all continuous functions from* K *to* $\mathbb{C}$*, endowed as usual with the sup-norm, i.e.,* $\|f\|_\infty = \sup_{x\in K}|f(x)|$*. Suppose* $(f_n)_{n\geqslant 1}$ *is a sequence in* $C(K)$*. Show that* $(f_n)_{n\geqslant 1}$ *converges weakly if and only if* $(f_n)_{n\geqslant 1}$ *is bounded and converges pointwise.*

Problem 9. *Let* $1 < p < q < r < \infty$ *and* $f\colon [0,1] \to \mathbb{R}$ *be measurable. Show that*

$$\|f\|_q \leqslant \|f\|_p^s \|f\|_r^{1-s}$$

where $s = \frac{\frac{1}{q}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{r}}$*.*

Problem 10.

(1) Give the definition of an orthonormal basis in a Hilbert space.

(2) Show that if a Hilbert space has a countably infinite orthonormal basis then every infinite orthonormal basis is countable.