Real Analysis Qualifying Exam, May, 2008

1. Prove that if E is a closed linear subspace of $L^2(0,1)$ and each element in E is bounded (i.e. $f \in L^\infty(0,1)$ for all $f \in E$), then E is finite dimensional.

2. Let $f_n := \sum_{k=1}^{2^n}(-1)^k \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n}]}$. Prove that $f_n \to 0$ weakly in $L^1(0,1)$.

3. Let E be a subset of a metric space X, and let $\mathcal{B} = \{\mathrm{B}(r(x), x) : x \in S\}$ be a collection of balls in X which cover E (that is, $E \subset \cup_{x \in S}\mathrm{B}(r(x), x)$) so that the radii are bounded (that is, $\sup\{r(x) : x \in S\} < \infty$). Prove that there is a (finite or infinite) sequence $\{\mathrm{B}(r(x_i), x_i)\}_{i=1}^N$ of disjoint balls in $\mathcal{B}$ so that either

(i) $N = \infty$ and $\inf\{r(x_i) : i = 1, 2, 3, \dots\} > 0$, or

(ii) $E \subset \cup_{n=1}^N \mathrm{B}(5r(x_i), x_i)$. ($N$ can be either finite or infinite in this case.)

4. Prove that every separable Banach space is isometrically isomorphic to a subspace of $C(\Delta)$, where Δ is the Cantor set. You may use the topological theorem that if X is a compact metric space, then there is a continuous surjection from Δ onto X.

5. For $f \in L^1(0,1)$ and $y \in [0,1]$, define $(Tf)(y) = \frac{1}{y}\int_0^y f(x)\,dx$. Show that T defines a bounded linear operator from $L^p(0,1)$ to $L^p(0,1)$ for all $1 < p \le \infty$, but T does not define a bounded linear operator from $L^1(0,1)$ to $L^1(0,1)$.

6. For $i = 1, 2$, let μ_i and ν_i be finite measures on a measurable space (X_l, Σ_i). Assume that $\mu_i << \nu_i$. Prove that $\mu_1 \times \mu_2 << \nu_1 \times \nu_2$.

7. Let $\{f_n\}_{n=1}^\infty$ be a sequence of non negative functions in $L^1(\mathbb{R})$ which converges pointwise almost everywhere to a function f in $L^1(\mathbb{R})$. Prove that if $\int_{\mathbb{R}} f_n(t)\,dt \to \int_{\mathbb{R}} f(t)\,dt$, then $\int_{\mathbb{R}} |f_n(t) - f(t)|\,dt \to 0$.

8. Let $f \in L^1(0,1)$ be such that for all $n = 1, 2, 3, \dots$, $\int_0^1 f(t)t^{2n}\,dt = 0$. Prove that $f = 0$ a.e.

9. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space X is said to be *weakly Cauchy* provided that for each $x^* \in X^*$, the sequence $\{x * (x_n)\}_{n=1}^\infty$ or real numbers is convergent. Let K be a compact Hausdorff space. Prove that a sequence $\{f_n\}_{n=1}^\infty$ in $C(K)$ is weakly Cauchy if and only if $\{f_n\}_{n=1}^\infty$ is bounded and pointwise convergent. Deduce that if $\{f_n\}_{n=1}^\infty$ is a weakly Cauchy sequence in $C[0,1]$, then $\{f_n\}_{n=1}^\infty$ is NORM convergent in $L^1(0,1)$.

10. Write μ for Lebesgue measure on the Borel subsets of the unit interval $[0,1]$. Recall that a Borel probability measure ν on $[0,1]$ is *singular* with respect to μ (in symbols, $\nu \perp \mu$) if there is a Borel set $D \subseteq [0,1]$ such that $\nu(D) = 1$ and $\mu(D) = 0$. Let ν be a Borel probability measure on $[0,1]$. Show that $\nu \perp \mu$ if and only if for every $\varepsilon > 0$ there is a continuous function $f : [0,1] \to [0,1]$ such that $\nu(f) > 1 - \varepsilon$ and $\mu(f) < \varepsilon$.