TOPICS COURSE IN GEOMETRY OF METRIC SPACES

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In this course selected topics of an emerging and fast-developing field, often refer to as *quantitative metric geometry* will be discussed. We will touch upon fundamental questions that originate from problems in theoretical computer science, group theory, topology, or theoretical physics. The common feature of all these problems is that they can be expressed in geometric terms. Building a geometric intuition will be central in our exposition. The course will be divided into three independent parts of roughly equal weight. All three parts are dealing with the general problem of "embedding faithfully" a certain type of metric space into a "nice" Banach space.

The first part is oriented towards applications in theoretical computer science and focuses on the quantitative theory of the bi-Lipschitz embeddability of finite metric spaces, in particular of finite graphs. It will be made accessible to anyone with a basic background of discrete mathematics, and discrete probabilities.

Part I: Low Distortion Embeddability of Finite Metric Spaces

1. Low distortion embeddings (stochastic decompositions of finite metric spaces, Bourgain's embedding)
2. Metric invariants (Enflo type, distortion lower bound for the Hamming cubes, Markov convexity, distortion lower bound for trees)
3. Dimension reduction (Johnson-Lindenstrauss reduction to logarithmic dimension in ℓ_2, dimension reduction lower bound in ℓ_1)
4. Expander graphs (spectral and combinatorial definitions, ℓ_p-distortion lower bound)
5. Application to the sparsest cut problem

The second part deals with the metric geometry of infinite metric spaces, in particular classical Banach spaces. Motivations from problems in topology and noncommutative geometry will be discussed.

Part II: Embeddability of Infinite Metric Spaces

1. Lipschitz embeddability of locally finite metric spaces (the gluing technique, Ostrovskii's finite determinacy theorem)
2. Large scale geometry of Banach spaces (snowflaking exponents, link beetwen the asymptotic structure of a Banach space and its coarse Lipschitz geometry, Kalton's property $\mathcal{Q}$)

The last part has a strong geometric group theoretic flavor but requires only a basic knowledge in group theory.

Part III: Large Scale Geometry of Finitely Generated Groups

1. Groups as geometric objects (left-invariant metrics on discrete countable groups, bounded geometry property of finitely generated groups, property A and exactness)
2. Compression theory (large Hilbertian compression implies exactness, random walks and compression exponent lower bounds)
3. Coarse embeddability of amenable groups (introduction to amenability, equivariant embeddings, a glimpse at the geometry of the discrete Heisenberg group)

Motivations. As networks and huge collection of data appear in so many different contexts, optimization problems in networks and data mining are becoming increasingly important. Quantitative geometry, in particular the theory of metric embeddings, received much attention in recent years by mathematicians as well as computer scientists, and has been applied in many algorithmic applications. The problem of finding sparse cuts in a graph is one variant of many graph problems where the goal is, loosely speaking, to find an edge cut with few edges that separates the graph into large components. Such problems have practical applications, for instance in the analysis of telecommunication networks, but also fundamental applications as a tool in the design of divide-and-conquer algorithms. Most variants of such problems are computationally extremely hard to solve, implying that it is unlikely to expect fast algorithms to provide exact solutions of the problems for large instances. In such cases one would like to design fast algorithms that guarantee approximate solutions close to the optimal solution. In the 90's, Linial, London and Rabinovich in a landmark article established the connection between the design of an approximation algorithm for the sparsest cut problem and the theory of bi-Lipschitz embeddings (in particular Bourgain's embedding theorem and the Johnson-Lindenstrauss lemma). This topic will be discussed in the first part of this course.

Since the early 1980's, A. Connes and others have been developing the subject of noncommutative geometry, a natural generalization of Riemannian geometry (much as Riemannian geometry, in turn, provides a natural generalization of Euclidean geometry). Its physical interest stems from the fact that it suggests an elegant geometric reinterpretation of the standard model of particle physics coupled to Einstein gravity. Part of Connes noncommutative program bridges classical geometry and topology, the best example arguably being the Baum-Connes conjecture which suggests that two objects associated to a group (an analytic one and a topological one) can be identified. The Baum-Connes conjecture "implies" many celebrated conjectures, in particular the Novikov conjecture. The latter conjecture, which says that the higher signatures (which are certain numerical invariants of smooth manifolds) are homotopy invariants, is one of the most important unsolved problem in topology. In his famous essay from 1993, M. Gromov discussed at length the efficiency of the geometric language for isolating interesting properties of groups. In 1995 Gromov hinted at that understanding the groups whose large-scale geometry is compatible in a certain sense with the geometry of a Hilbert space or a superreflexive (i.e., admits an equivalent uniformly convex norm) Banach space should be interesting regarding the Novikov conjecture. Building upon a groundbreaking work of G. Yu, Gromov's intuition was eventually proved to be true in the mid-2000's by Kasparov and Yu. The attention was then drawn to the coarse geometry of Banach spaces, which contrary to its uniform counterpart, had been little considered up to that time. The large scale geometry of infinite metric spaces, in particular groups, is the common ground of the second and third part of this course.

Bibliography

1. P. Nowak and G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zŭrich, 2012.
2. M. I. Ostrovskii, *Metric Embeddings*, De Gruyter Studies in Mathematics, vol. 49, De Gruyter, Berlin, 2013. Bilipschitz and coarse embeddings into Banach spaces.