2015 TAMU High School Math Contest

Power Team Exam

Subcollection Sum Divisibility Theorems

Solutions

The goal of this problem set is to determine the values of k and n for which the following statement is true or false:

$P(k,n)$: For every collection, S, of n integers there is a subcollection, T, of k integers whose sum is divisible by k.

The first group of problems give examples with values of k and n where $P(k,n)$ is true. The second group of problems will determine all values of k and n for which $P(k,n)$ is false by giving counterexamples. And the third group of problems will determine all values of k and n for which $P(k,n)$ is true. First some definitions.

Definitions: A *collection* is a list of elements in which elements are allowed to repeat and the order does not matter. So for example, $S = \{2,3,2,5,4,7,3\}$ is a collection of 7 numbers. Since the order of the elements does not matter, we can also write $S = \{2,2,3,3,4,5,7\}$ where we have written the numbers in ascending order. The number of times an element repeats is its *multiplicity*. A collection T is a *subcollection* of S (written $T \subseteq S$) if every element of T is an element of S and the multiplicity of each element in T is less than or equal to its multiplicity in S. Thus for the same example, $\{2,2,3,5\} \subseteq S$ but $\{2,3,3,3,4,5\} \not\subseteq S$. We write $|S|$ for the number of elements in S and ΣS for the sum of the elements in S. So for the example S above, we have $|S| = 7$ and $\Sigma S = 26$. We also define $S + r$ to be the collection where r has been added to each element of S, and rS to be the collection where each element of S has been multiplied by r. For the example S above, we have $S + 3 = \{5,5,6,6,7,8,10\}$ and $3S = \{6,6,9,9,12,15,21\}$.

Rules: In proving each statement you can use the results of previous statements, even if you have not been able to prove them. However, you can only use the results of subsequent statements if you actually prove them.

Problem Group 1:

In this group of problems, you will determine several values of k and n for which $P(k,n)$ is true.

1. **Prove** $P(2,3)$: For every collection, S, of 3 integers there is a subcollection, T, of 2 integers whose sum is divisible by 2.

Proof: Let $S = \{a,b,c\}$. If 2 or 3 of a,b,c are even, then the sum of 2 even elements is even and divisible by 2. If 0 or 1 of a,b,c are even, then there are two odd elements which sum to an even number which is dividible by 2. ■

2. **Lemma**: If T is a collection of k integers which are all equal to each other $\bmod k$ (that is, they all have the same remainder when divided by k), then the sum of the elements in T is divisible by k.

Proof: The elements of T satisfy $a_1 \equiv a_2 \equiv \cdots \equiv a_k \equiv r \bmod k$. So $a_1 + a_2 + \cdots + a_k \equiv kr \equiv 0 \bmod k$. ■

3. **Lemma**: If T is a collection of k integers, then the sum $\bmod k$ of the elements in $T+r$ is equal to the sum $\bmod k$ of the elements in T.

In other words, the sum of the elements in $T+r$ and the sum of the elements in T have the same remainder when divided by k. Consequently, if the sum of the elements in T is divisible by k, then the sum of the elements in $T+r$ is also divisible by k and vice versa.

Proof: If $T = \{a_1, a_2, \cdots, a_k\}$ then $T+r = \{a_1+r, a_2+r, \cdots, a_k+r\}$ and $(a_1+r)+(a_2+r)+\cdots+(a_k+r) = a_1+a_2+\cdots+a_k+rk \equiv a_1+a_2+\cdots+a_k \bmod k$. ■

4. **Prove** $P(4,7)$: For every collection, S, of 7 integers there is a subcollection, T, of 4 integers whose sum is divisible by 4.

Proof: Assume the elements of S are ordered in ascending order $\bmod 4$.

By Lemma 2, if 4 or more of the elements of S are congruent to the same number, say $r \bmod 4$, then the sum of 4 of those is divisible by 4.

By Lemma 3, if you add the same number, say r, to all 7 numbers, then the sum of any 4 numbers does not change $\bmod 4$.
Observe that each of the 7 numbers is equal to one of the numbers $0,1,2,3 \bmod 4$. By the pigeonhole principle, one of the numbers $0,1,2,3 \bmod 4$ must occur at least twice.

Consequently, if we identify which number ($\bmod 4$) occurs most often and subtract that from all 7 numbers, then $0 \bmod 4$ will occcur most often which is at least twice.

So we can assume that each of the numbers $0,1,2,3 \bmod 4$ occurs at most 3 times and 0 occurs at least twice and occurs most often. These are the remaining possible values for $S \bmod 4$, and how the subcollection T of 4 elements is chosen:

$S \equiv \{0,0,0,1,1,1,2\}$	$T \equiv \{0,1,1,2\}$	$S \equiv \{0,0,0,1,3,3,3\}$	$T \equiv \{0,0,1,3\}$
$S \equiv \{0,0,0,1,1,1,3\}$	$T \equiv \{0,0,1,3\}$	$S \equiv \{0,0,0,2,2,2,3\}$	$T \equiv \{0,0,2,2\}$
$S \equiv \{0,0,0,1,1,2,2\}$	$T \equiv \{0,0,2,2\}$	$S \equiv \{0,0,0,2,2,3,3\}$	$T \equiv \{0,0,2,2\}$
$S \equiv \{0,0,0,1,1,2,3\}$	$T \equiv \{0,0,1,3\}$	$S \equiv \{0,0,0,2,3,3,3\}$	$T \equiv \{0,2,3,3\}$
$S \equiv \{0,0,0,1,1,3,3\}$	$T \equiv \{0,0,1,3\}$	$S \equiv \{0,0,1,1,2,2,3\}$	$T \equiv \{0,0,1,3\}$
$S \equiv \{0,0,0,1,2,2,2\}$	$T \equiv \{0,0,2,2\}$	$S \equiv \{0,0,1,1,2,3,3\}$	$T \equiv \{0,0,1,3\}$
$S \equiv \{0,0,0,1,2,2,3\}$	$T \equiv \{0,0,2,2\}$	$S \equiv \{0,0,1,2,2,3,3\}$	$T \equiv \{0,0,1,3\}$
$S \equiv \{0,0,0,1,2,3,3\}$	$T \equiv \{0,2,3,3\}$		

In each case, $\Sigma T \equiv 0 \bmod 4$. ■

5. **Theorem**: If $n = k^2-k+1$ then $P(k,n)$ is true.

Proof: We apply the pigeonhole principle to distribute the n numbers in S into the k values $0,1,2,\cdots,k-1 \bmod k$. If we have k numbers with the same value $\bmod k$, we use them as the subcollection T by Lemma 2. To avoid this, we can assign at most $k-1$ numbers to each of the k values using up $k(k-1) = k^2-k$ numbers. However, since $n = k^2-k+1$ at least one value $\bmod k$ must be repeated k times and therefore the subcollection T will exist. ■

Problem Group 2:

In this group of problems, you will determine all values of k and n for which $P(k,n)$ is false.

6. **Counterexample to** $P(2,2)$: Find a collection, S, of 2 integers for which there is no subcollection, T, of 2 integers whose sum is divisible by 2.

Solution: Let $S = \{0,1\}$. The only subcollection of 2 integers is $\{0,1\}$ and $0 + 1 = 1$ which is not divisible by 2. ■

7. **Counterexample to** $P(4,6)$: Find a collection, S, of 6 integers for which there is no subcollection, T, of 4 integers whose sum is divisible by 4.

Solution: Let $S = \{0,0,0,1,1,1\}$. The only subcollections of 4 integers and their sums are:

$\{0,0,0,1\}$ and $0 + 0 + 0 + 1 = 1$ which is not divisible by 4
$\{0,0,1,1\}$ and $0 + 0 + 1 + 1 = 2$ which is not divisible by 4
$\{0,1,1,1\}$ and $0 + 1 + 1 + 1 = 3$ which is not divisible by 4 ■

8. **Counterexample to** $P(k,2k-2)$: Find a collection, S, of $2k-2$ integers for which there is no subcollection, T, of k integers whose sum is divisible by k.

Solution: Let $S = \{0,\cdots,0,1,\cdots,1\}$, where there are $k-1$ values of 0 and $k-1$ values of 1. The only subcollections of k integers and their sums are:

$\{0,\cdots,0,1,\cdots,1\}$, where there are r values of 0 and $k-r$ values of 1 with $1 \le r \le k-1$

The sum is $r \bullet 0 + (k-r) \bullet 1 = k - r$ which is not divisible by k. ■

9. **Lemma**: If $P(k,n)$ is true for some k and n, then $P(k,m)$ is also true for any $m > n$.

Proof: Assume $P(k,n)$ is true for some k and n and let $m > n$. Let S be a collection with m elements. Then let S' be a subcollection of S with n elements. By $P(k,n)$, there is a subcollection, T, of S' consisting of k integers whose sum is divisible by k. Then T is also a subcollection of S with the same properties. ■

10. **Theorem**: For every k, if $n \le 2k-2$, then $P(k,n)$ is false.

Proof: If $n = 2k-2$, then problem 8 provides a counter example to $P(k,n)$.

If $n < 2k-2$, we use proof by contradiction. Assume $P(k,n)$ is true. Then by Lemma 9, $P(k,2k-2)$ is true. However, problem 8 provides a counterexample to $P(k,2k-2)$. So $P(k,n)$ must be false.
■

Problem Group 3:

In this group of problems, you will determine all values of k and n for which $P(k,n)$ is true. The strategy is to prove $P(k,2k-1)$ when k is prime and separately when k is a product of two numbers for which the theorem has already been proved. An application of mathematical induction will give the theorem for all k's. Since the second part is easier, we prove that first:

11. **Theorem**: If $P(r,2r-1)$ and $P(s,2s-1)$ are true, then $P(rs,2rs-1)$ is true.

HINT: If S is a collection of $2rs-1$ numbers, construct disjoint subcollections $T_1,T_2,\cdots,T_{2s-1}$ each with r elements whose sum is divisible by r.

Proof: We start with the collection S of $n = 2rs-1$ numbers. Arbitrarily select $2r-1$ of these and call it the collection $S_1 \subseteq S$. By $P(r,2r-1)$, there is a subcollection T_1 with r numbers whose sum is divisible by r. Let this sum be $\Sigma T_1 = t_1 r$ and define $S_1' = S - T_1$. In other words, we remove the r elements of T_1 from the collection S to produce the collection S_1' which has $2rs-1-r$ elements. We repeat this process, arbitrarily picking a subcollection $S_2 \subseteq S_1'$ with $2r-1$ numbers. Within S_2 there is a subcollection T_2 with r numbers whose sum is $\Sigma T_2 = t_2 r$. Then we define $S_2' = S_1' - T_2$ with $2rs-1-2r$ elements. We want to repeat this process $2s-1$ times but we need to check the last step works properly. So suppose we have already produced sets $T_1,T_2,\cdots,T_{2s-2}$ each with r elements whose sums are $t_1 r, t_2 r,\cdots,t_{2s-2}r$ leaving behind the collection S_{2s-2}' with $2rs-1-(2s-2)r = 2r-1$ elements. This is the exact number we need to apply $P(r,2r-1)$ one last time producing the collection T_{2s-1} with r numbers whose sum is $\Sigma T_{2s-1} = t_{2s-1}r$.
Next, let $\bar{S} = \{t_1,t_2,\cdots,t_{2s-2},t_{2s-1}\}$. We apply $P(s,2s-1)$ to the collection $\bar{S}$ to produce a subcollection $\bar{T} = \{t_{i_1},t_{i_2},\cdots,t_{i_s}\}$ with s elements whose sum is divisible by s. Then the sum of the numbers in $r\bar{T}$ is divisible by rs. Finally, let $T = T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_s}$. Then T has rs elements whose sum is divisible by rs. ■

We next prove $P(p,2p-1)$ is true for primes p. First a lemma:

12. **Lemma**: Assume p is prime. Let A be a subset of $I = \{0,1,2,\cdots,p-1\}$ with n distinct elements where $1 \le n < p$ and let B be a subset of I with 2 distinct elements. Define $(A+B)\bmod p$ to be the set $\{(a+b)\bmod p \mid a \in A \text{ and } b \in B\}$. Then $(A+B)\bmod p$ is a subset of I with at least $n+1$ distinct elements.

Proof: Let $A = \{a_1,a_2,\cdots,a_n\}$ where $a_1 < a_2 < \cdots < a_n$ and $B = \{b_1,b_2\}$ where $b_1 < b_2$. Since the numbers $a_1,a_2,\cdots,a_n$ are distinct, the numbers $(a_1+b_1),(a_2+b_1),\cdots,(a_n+b_1)\bmod p$ are also distinct. So the set $(A+B)\bmod p$ has at least n distinct elements. To get a proof by contradiction, assume that $(A+B)\bmod p$ has only n distinct elements. Then the distinct numbers $(a_1+b_2),(a_2+b_2),\cdots,(a_n+b_2)\bmod p$ would have to coincide with the numbers $(a_1+b_1),(a_2+b_1),\cdots,(a_n+b_1)\bmod p$ but possibly in a different order. Adding the elements in each set we get

$$a_1+a_2+\cdots+a_n+nb_1 = a_1+a_2+\cdots+a_n+nb_2 \bmod p$$

So $nb_1 = nb_2 \bmod p$. Since p is prime and $1 \le n < p$, we have $b_1 = b_2 \bmod p$ which contradicts the hypothesis that B has 2 distinct elements. Therefore, $(A+B)\bmod p$ has at least $n+1$ distinct elements. ■

13. **Theorem**: If p is prime, then $P(p, 2p-1)$ is true.

HINT: Let $S = \{a_1, a_2, \cdots, a_{2p-1}\}$ be the collection of $2p-1$ numbers written in ascending order: $a_1 \le a_2 \le \cdots \le a_{2p-1} \bmod p$. Explain why you can assume no more than $p-1$ of these are equal. Construct T to contain the number a_1 plus one number from each pair $\{a_2, a_{p+1}\}$, $\cdots$, $\{a_i, a_{i+p-1}\}$, $\cdots$, $\{a_p, a_{2p-1}\}$. You need to show that the numbers can be chosen from each pair so that $\Sigma T = 0 \bmod p$. Use the lemma.

Proof: Let $S = \{a_1, a_2, \cdots, a_{2p-1}\}$ be a collection of $2p-1$ numbers written in ascending order: $a_1 \le a_2 \le \cdots \le a_{2p-1} \bmod p$. Let $A_1 = B_1 = \{a_1\}$. Let $B_i = \{a_i, a_{i+p-1}\}$ for $i = 2, \cdots p$ and recursively let $A_i = (A_{i-1} + B_i) \bmod p$. First notice that we can assume that the numbers in each pair $B_i = \{a_i, a_{i+p-1}\}$ are distinct. Otherwise the p numbers $a_i \le \cdots \le a_{i+p-1}$ would all be equal and would form the subcollection T which proves the theorem by Lemma 2. Now since $|A_1| = 1$, by Lemma 12, $|A_2| \ge 2$. Recursively, Lemma 12 implies $|A_i| \ge i$. In particular, $|A_p| \ge p$. However, there are only p numbers $\bmod p$. So $|A_p| = p$ and $A_p = \{0, 1, \cdots, p-1\}$. Thus $0 \in A_p$. However, from the recurrsive definition of the sets A_i, it follows that $A_p = B_1 + B_2 + \cdots + B_p \bmod p$. So $0 = b_1 + b_2 + \cdots + b_n \bmod p$ for some $b_i \in B_i$. Let $T = \{b_1, b_2, \cdots, b_n\}$. Then T is a subcollection of S with n elements and $\Sigma T = 0 \bmod p$. ■

14. **Theorem**: If k is a positive integer ≥ 2, then $P(k, 2k-1)$ is true.

Proof: We apply strong mathematical induction. First for the initialization step, we know $P(2,3)$ is true by problem 1. For the induction step, we assume $P(k, 2k-1)$ is true for all $k < K$ and prove $P(K, 2K-1)$. If K is prime, then $P(K, 2K-1)$ is true by Theorem 13. If K is not prime, then $K = rs$ for some $r < K$ and $s < K$. Then $P(K, 2K-1)$ is true by Theorem 11. So $P(k, 2k-1)$ is true for all k.
■

15. **Corollary**: If $n \ge 2k-1$, then $P(k, n)$ is true.

Proof: By Theorem 14 and Lemma 9, $P(k, n)$ is true. ■

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