2022 Power Team, Solutions
Texas A&M High School Students Contest
November 2022

Placing points into shapes, cutting pies, and escaping shapes

1. A set of points is located on the plane so that any triangle with vertices at these points has area at most 1. Prove that all these points lie in a triangle of area 4 (when we say "lie in a triangle" we mean that they may also lie on its boundary).

   Solution Among all triangles with vertices in the set consider a triangle $\triangle A_1A_2A_3$ of maximum area $s \leq 1$. Through every vertex of this triangle draw the line parallel to the opposite side of the triangle. The area of the triangle $\triangle B_1B_2B_3$ created by these lines is equal to $4s \leq 4$, see Figure 1 .

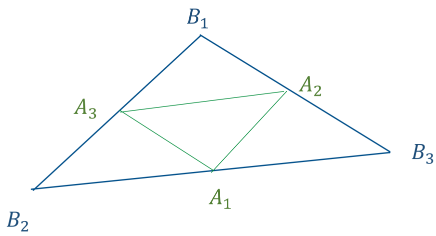

Figure 1

   We claim that all points of the set lie inside of this triangle. For this note that a point X satisfies the condition that the area of triangle A_1A_2X is not greater than the area of triangle $A_1A_2A_3$ if and only if X lies in the strip between the line B_1B_2 and the line parallel to B_1B_2 and passing through B_3. Similarly, a point X satisfies the condition that the area of triangle A_2A_3X is not greater than the area of triangle $A_1A_2A_3$ if and only if X lies in the strip between the line B_2B_3, and the line parallel to B_2B_3 and passing through B_1 and a point X satisfies the condition that the area of triangle A_1A_3X is not greater than the area of triangle $A_1A_2A_3$ if and only if X lies in the strip between the line B_1B_3 and the line parallel to B_1B_3 and passing through B_2. Since by construction $\triangle A_1A_2A_3$ is the triangle with the maximal area among all triangles with vertices at the set, all other points of the set must belong to the intersection of the aforementioned three strips. This intersection is equal to the triangle $\triangle B_1B_2B_3$, which has the area $4s \leq 4$, q.e.d.

2. (a) Is it possible to choose 6 points in the disc of radius 1 such that the distance between any two of them is greater than 1? Prove your answer.

   (b) Is it possible to choose 441 points in the disc of radius 10 such that the distance between any two of them is greater than 1? Prove your answer.

   Solution

(a) The answer is **no**. Assume by contradiction that there is a configuration of 6 points $A_1, \dots, A_6$ inside of the unit disc of radius 1 with center O. First, note that O is not one of these points, as all points of the disc are on the distance at most 1 from O. On each ray OA_i there are no points A_j other than A_i. Then there exist $i_1 < i_2$ such that the angle $\angle A_{i_1}OA_{i_2}$ is not greater than $60°$. Since $|OA_i| \leq 1$ this implies that $|A_{i_1}A_{i_2}| \leq 1$, which leads to the contradiction.

(b) The answer is **no**. Assume that in a disc of radius 10 centered at a point O there exist N points such that the distance between any two of them is greater than 1. Then the disks of radius $\frac{1}{2}$ centered at these points are disjoint (any two of them have an empty intersection) and all such discs lie in the disk of radius 10.5 centered at O. Moreover the complement of the union of those smaller disc to the larger disc clearly has positive area. Consequently, the sum of the areas of those discs is less than the area of the disc of radius 10.5, i.e. $\pi\dfrac{N}{4} < \pi(10.5)^2 = \pi\dfrac{441}{4}$, i.e. $N < 441$. This completes the proof of our answer.

3. A pie has a shape of a regular n-gon inscribed into a circle of radius 1 (we assume that the pie is planar, ignoring its thickness). From the middle of each edge of this n-gon in an arbitrary direction one makes a cut along a segment of length at least 1. Prove that after these cuts a piece is cut off from the pie.

Solution

Let $A_1, A_2 \dots, A_n$ be the vertices of the pie, O be its center, $M_1, M_2, \dots, M_n$ be the middle points of the edges , and M_1N_1, M_2N_2, $\dots$, M_nN_n be the cuts (see Figure 2.)

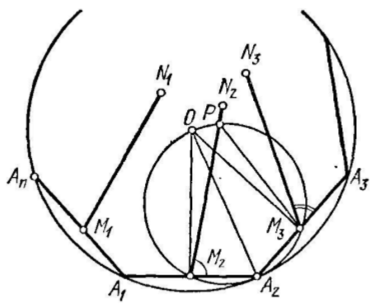

Figure 2

Assume that the statement of the problem is wrong. For definiteness assume that $\angle A_2M_2N_2 < 90°$. Construct a circle C with diameter OA_2. Since $\angle OM_2A_2 = \angle OM_3A_3 = 90°$, points M_2 and M_3 lie on this circle. Since $|M_2N_2| \geq 1$ and $|OA_2| = 1$, the point N_2 cannot lie inside of the constructed circle C. Let P the point of intersection of the cut M_2N_2 with C (P may coincide with N_2). Then

$$\angle PM_2A_2 = \angle PM_3A_3, \tag{1}$$

Indeed , since the quadrilateral $PM_2A_2M_3$ is inscribed in C , $\angle PM_2A_2 = 180° - \angle PM_3A_2$ but $180° - \angle PM_3A_2 = \angle PM_3A_3$, which completes the proof of (1).

As a consequence $\angle A_2M_2N_2 = \angle A_3M_3P > \angle A_2M_2N_2$, otherwise, if the last inequality does not hold, the cuts M_2N_2 and M_3N_3 intersect, which will cut off a piece from the pie.

In the same way we can prove the following chain of strict inequalities

$$\angle A_2M_2N_2 > \angle A_3M_3N_3 > \angle A_4M_4N_4 > \dots \angle A_nM_nN_n > \angle A_1M_1N_1 > \angle A_2M_2N_2,$$

so that at the end we get $\angle A_2M_2N_2 > \angle A_2M_2N_2$ that is a contradiction.

4. (a) A pie has a shape of a square of side length 1 (again we assume that the pie is planar, ignoring its thickness). Divide each edge of this pie into 3 equal pieces by marking two corresponding points on each edge. Cut from the square four corner triangles which are obtained as follows: for every such triangle one vertex coincides with a vertex of the pie and two other vertices are the marked points on the adjacent edges closest to this vertex. Then repeat the process with the resulting octagonal pie: divide each edge of this octagon into 3 equal pieces by marking two corresponding points on each edge, then cut from the octagon eight triangles which are obtained as follows: for every such triangle one vertex coincides with a vertex of the octagon and two other vertices are the marked points on the adjacent edges closest to this vertex. Find the area of the pie obtained by repeating this procedure infinitely many times.

   (b) Solve the same problem if on each step the edge is divided into p equal pieces with $p > 2$ (as in the previous item for any triangle removed on the nth step one vertex coincides with a vertex of the polygon of the previous $(n-1)$st step and two other vertices are the marked points on the adjacent edges closest to this vertex).

Solution Let P_0 is the original pie and P_k is the pie obtained on the kth step. Let s_k be the area of the pie P_k.

(a) The answer is $\boxed{\frac{5}{7}}$.

Let us prove that

$$s_k - s_{k+1} = \frac{2}{9}(s_{k-1} - s_k). \qquad (2)$$

For this it is enough to prove the following

Lemma 1. *The area of the one corner triangle removed on the $(k+1)$st step is equal to $\frac{1}{9}$ of the area of the corner triangle removed on the kth step.*

Lemma 1 implies (2), because the number of triangles removed in the $(k+1)$st step is twice of the number of triangles removed on the kth step. To prove Lemma 1 note that if $\triangle ABC$ and $\triangle BDE$ are two adjacent corner triangles that are cut on the kth and $(k+1)$st steps , respectively,

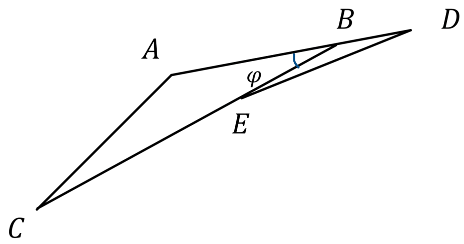

Figure 3

then

$$\frac{|BD|}{|BA|} = \frac{1}{3}, \quad \frac{|BE|}{|BC|} = \frac{1}{3}, \tag{3}$$

see Figure 3. Hence,

$$\frac{\text{Area}(\triangle BDE)}{\text{Area}(\triangle ABC)} = \frac{1/2|BD||BE|\sin(180^\circ - \varphi)}{1/2|BA||BC|\sin\varphi} = \frac{1}{9},$$

which completes the proof of the lemma and hence of (1).

Note that $s_0 - s_1 = 4\left(\frac{1}{2}\right)\frac{1}{3^2} = \frac{2}{9}$. Therefore (1) implies that

$$s_{k-1} - s_k = \left(\frac{2}{9}\right)^k,$$

i.e. the sequence $\{s_{k-1} - s_k\}$ forms a geometric progression.

Using the telescopic sum and the formula for the sum of geometric progression, we get

$$s_k = s_0 + (s_1 - s_0) + (s_2 - s_1) + \ldots + (s_k - s_{k-1}) = 1 - \frac{2}{9} - \left(\frac{2}{9}\right)^2 - \ldots - \left(\frac{2}{9}\right)^k =$$

$$1 - \frac{2}{9}\,\frac{1 - (2/9)^{k+1}}{1 - 2/9} \xrightarrow[k\to\infty]{} 1 - \frac{2}{9}\,\frac{1}{1 - 2/9} = 1 - \frac{2}{7} = \boxed{\frac{5}{7}}.$$

(b) The answer is $\boxed{\dfrac{p^2 - 2p + 2}{p^2 - 2p + 4}}$. The solution is similar to item (a). In this case (2) is replaced by

$$s_k - s_{k+1} = \frac{2p - 4}{p^2}(s_{k-1} - s_k). \tag{4}$$

and the latter is obtained from

Lemma 2. *The area of the one corner triangle removed on the* $(k+1)$*st step is equal to* $\frac{p-2}{p^2}$ *of the area of the corner triangle removed on the* k*th step.*

To prove the lemma note that (3) is replaced by

$$\frac{|BD|}{|BA|} = 1 - \frac{2}{p}, \quad \frac{|BE|}{|BC|} = \frac{1}{p} \tag{5}$$

(see Figure 4).

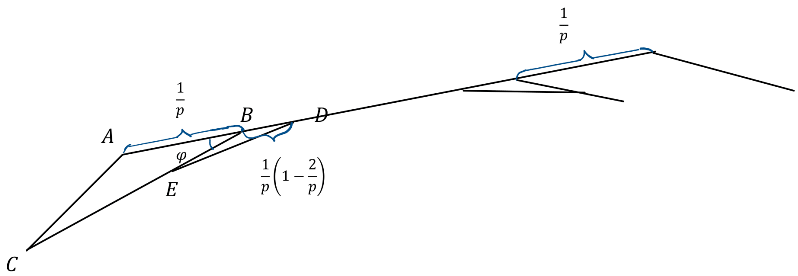

Figure 4

so that the ratio of areas of the triangles is equal to $\frac{1}{p}\left(1 - \frac{2}{p}\right) = \frac{p-2}{p^2}$. Further, $s_0 - s_1 = \frac{2}{p^2}$, which implies that

$$s_k - s_{k+1} = \frac{2}{p^2}\left(\frac{2p-4}{p^2}\right)^k.$$

Using telescopic sum and the formula for the sum of geometric progression

$$\begin{aligned}
&s_k = s_0 + (s_1 - s_0) + (s_2 - s_1) + \ldots + (s_k - s_{k-1}) = \\
&1 - \frac{2}{p^2}\left(1 + \frac{2p-4}{p^2} + \left(\frac{2p-4}{p^2}\right)^2 + \ldots + \left(\frac{2p-4}{p^2}\right)^{k-1}\right) = \\
&1 - \frac{2}{p^2}\,\frac{1 - \left(\frac{2p-4}{p^2}\right)^k}{1 - \frac{2p-4}{p^2}} \xrightarrow[k\to\infty]{} 1 - \frac{2}{p^2}\,\frac{1}{1 - \frac{2p-4}{p^2}} = 1 - \frac{2}{p^2 - 2p + 4} \\
&= \boxed{\frac{p^2 - 2p + 2}{p^2 - 2p + 4}}.
\end{aligned}$$

(in the passage to the limit we used that $0 < \frac{2p-4}{p^2} < 1$ for $p > 2$: the left inequality is obvious from assumptions, the right ineqiuality follows from $1 - \frac{2p-4}{p^2} = \frac{p^2-2p+4}{p^2} > \left(\frac{p-1}{p}\right)^2 > 0$.)

5. A hare sits in the center of a square, and one wolf sits in each of the four corners. Wolves can only run along the borders of the square, and the hare can move in any direction. At each moment the wolves and the hare know the location of all participants and there is no delay in the reaction of any participants on the change of location of any other.

   (a) Can the hare run out of the square without being caught by at least one wolf if the maximum speed of the wolves is 1.4 times greater than the maximum speed of the hare? Prove your answer.

   (b) At what minimum ratio r^* of the maximum speeds of the wolves and the hare do wolves always have a strategy in which they can catch a hare? Prove your answer.

Solution

(a) The answer is **yes**. WLOG assume that the side of the square has length 1. Assume that A_1, A_2, A_3, and A_4 are the vertices of the square, say in counterclockwise order, and the ith wolf, called W_i , was in the vertex A_i at the initial time moment. We will search for a winning escaping strategy of the hare among the following ones: First he runs the distance of s, where $0 < s < \frac{1}{\sqrt{2}}$ with his maximal velocity along the diagonal toward one of the vertices of the square, say A_1. Then he makes a turn of $90°$ and continues with its maximal velocity toward the edge with vertex A_1, which does not contain the wolf W_1 inside of it at the time of hare's turn (if at this time moment the wolf W_1 is located at the vertex A_1, then it does not matter toward which of the edges A_1A_2 or A_1A_3 the hare will make its turn). Our claim is that, with this strategy, for s sufficiently close to $\frac{1}{\sqrt{2}}$ the hare will always escape the square. Indeed, assume WLOG that the hare made its turn toward A_1A_2 and he reaches A_1A_2 at a point B. Then

$$|A_1B| = \sqrt{2}(\frac{1}{\sqrt{2}} - s) > 1.4(\frac{1}{\sqrt{2}} - s). \tag{6}$$

Further, the distance the hare run after his turn is equal to $\frac{1}{\sqrt{2}} - s$. Hence, the maximal distance the wolf W_1 can run after the hare's turn is $1.4(\frac{1}{\sqrt{2}} - s)$. Beside, by our assumptions, the wolf W_1 at the time of the hare's turn is on the other edge A_1A_3. Therefore, by the inequality in (6) the wolf W_1 is unable to reach the hare at the point B.

On the other hand, by the equality in (6), we have that $|A_2B| = 1 - |A_1B| = \sqrt{2}s$. Since before reaching B he travels total distance of $\frac{1}{\sqrt{2}}$ the maximal distance that the wolf W_2 can run by this time moment is $\frac{1.4}{\sqrt{2}}$. So, the wolf W_2 is unable to reach the hare at the point B if $|A_2B| = \sqrt{2}s > \frac{1.4}{\sqrt{2}}$. i.e., if $s > \frac{1.4}{2}$. So, for all $s \in \left(\frac{1.4}{2}, \frac{1}{\sqrt{2}}\right)$ the described escaping strategy of the hare is winning.

(b) The answer is $\boxed{r^* = \sqrt{2}}$. First, repeating exactly the same argument as in the proof of the previous item it can be shown that the strategy described there is the hare's winning escaping strategy for any ratio r of maximal velocity of the wolves and the hare with $r < \sqrt{2}$, just take $s \in \left(\frac{r}{2}, \frac{1}{\sqrt{2}}\right)$ there.

Now assume that $r = \sqrt{2}$. Given two nonparallel straight lines b and ℓ in the plane let $\pi_{b,\ell}$ be the projection to the line b by the lines parallel to ℓ, i.e. $\pi_{b,\ell}(p)$ is the point of intersection of the line b with the line passing through a point p parallel to ℓ (see Figure 5 below).

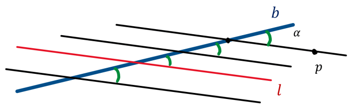

Figure 5

The following lemma is crucial for this and the next problem:

Lemma 3. *Let b and ℓ are nonparallel straight lines and α is the angle between them so that $\alpha \in (0, \frac{\pi}{2}]$. If an object moves along a trajectory $p(t)$ in $\mathbb{R}^2$ then at every time moment the ratio of the speed of the projection $\pi_{b,\ell}(p(t))$ of this object to the line b by lines parallel to the line ℓ and the speed of the object itself is not greater than $\frac{1}{\sin \alpha}$. The maximum of the ratio is achieved when the velocity of the object is orthogonal to ℓ.*

The proof is demonstrated by the following Figure 6:

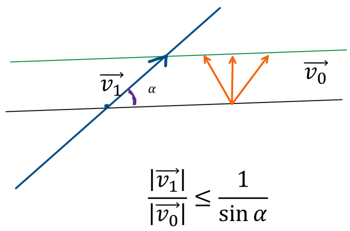

Figure 6

If at a given time moment t the hare is at a point $p(t) = (x(t), y(t))$, then consider the cross formed by two lines passing through $p(t)$ and parallel to the diagonals of the square. Assume that the group of four wolves moves such that at a time moment $t \geq 0$ they are located in the points of intersection of this cross with the boundary of the square. In fact, if O is the center of the square, then the trajectory of the wolf W_i is the continuous trajectory obtained by the projection of the trajectory of the hare to the corresponding edge of the square by lines parallel to the line OA_i. Since the angles between the line OA_i and the edges of the square are equal to $45°$ by Lemma 3 the speed of every wolf is not greater than $\sqrt{2}$ of the speed of the hare, so the motion of the group of wolves does not violate the restriction for the velocity. With this strategy the hare arrives to the boundary of the square together with at least two wolves so the wolves have the winning strategy to catch the hare.

6. Assume that in the setting of the previous problem a moose tries to escape the square instead of a hare. A moose is strong enough so that one wolf cannot hold him, but two wolves can.

   (a) At what minimum ratio r^* of the maximum speeds of the wolves and the moose do wolves always have a strategy with which they can catch the moose? Here, as in Problem 5, we assume that the moose starts at the center of the square and the wolves start at the corners. Prove your answer.

   (b) Assuming that the ratio of the maximum speeds of the wolves and the moose is $r > r^*$, where r^* is the critical ratio from part (a), find the set of all initial positions of the moose inside the square such that the wolves always have a strategy with which they can catch the moose (if they start at the corners).

Solution (a) The answer is the same as in Problem 5 (b) i.e. $\boxed{r^* = \sqrt{2}}$ as for $r < \sqrt{2}$ the moose has the winning escaping strategy and for $r^* > \sqrt{2}$ in the wolves's winning catching strategy at least two volves arrive to the point when the moose reaches the boundary of the square at the same point.

(b) The answer can be described as follows. From every vertex of the triangle draw two rays that constitute angle $\alpha = \arcsin \dfrac{1}{r}$ with the edges adjacent to this edge and consider the wedge (the infinite sector) between those two rays. Then the desired set is the octagon obtained by the intersection of these 4 wedges or by the intersection of two squares , whose boundaries are formed by 4 out of 8 rays having angle α with the edges counterclockwise for all of them or clockwise for all of them, see Figure 3, where also the dynamics of the octagon is shown as the angle α decreases.

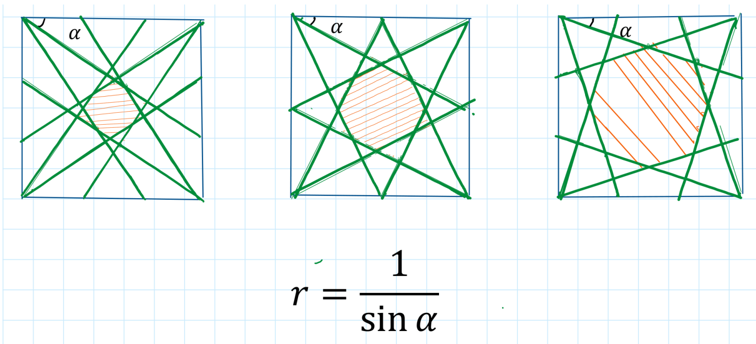

Figure 7

Indeed this octagon can be characterized as follows: a point C belongs to the octagon if the minimal angle β between the segment CA_i , where A_i is a vertex of the square) and the edges of the square is not less than α.

Assume that the moose starts at a point C of the octagon. Connect the point C with each vertex and apply the analogous strategy here as in solution of problem 5(b): the location of the wolf W_i at a given time moment is obtained from the location of the hair at the same time moment by applying the projection by lines parallel to the line CA_i to the corresponding edge of the square so that the trajectory of the wolf is continuous. Since the angles of the lines CA_i with the edges of the square are not less than α then by Lemma 3 the ratio of speed of the wolves and the moose at every time moment will be not greater than $\frac{1}{\sin\alpha} = r$, so this strategy of wolves satisfies the speed ratio restrictions. With this strategy the moose arrives to the boundary of the square together with at least two wolves so the wolves have the winning strategy to catch the moose.

Now, if the moose starts at a point C outside of the octagon, then C is strictly inside of one of the right triangles with hypotenuse coinciding with the edge of the square and with one of the angles equal to α, as in the example on Figure 8.

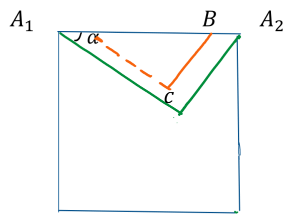

Figure 8

Then draw the right triangle with the right angle at the starting point C and the legs parallel to the triangle on figure 8. If the moose will move with maximal velocity along the leg CB opposite to the angle α, then the wolf from the vertex A_1 of the angle α in the Figure will run the distance not greater than the length of hypothenuse of the smaller triangle, and so he will not be able to arrive to the point B of the intersection of the moose with the edge and the only wolf, who is able to arrive to B, is the one starting from the other vertex A_2 of the same edge.

7. An island in a sea is so small that its size can be ignored and it can be assumed to be a point. A light house is installed on the island. At every time moment the beam of the projector can illuminate a narrow sector of the sea surface of length L (the thickness of the sector is ignored). The projector rotates uniformly around the vertical axis (in a fixed direction), making one revolution per time interval T. The boat, which can move at a speed v, must approach the island imperceptibly (that is, without falling into the searchlight beam).

(a) Can the boat succeed if its maximal velocity v is $0.9\dfrac{L}{T}$?

(b) Find the minimal v_0 such that if $v > v_0$ the boat can succeed? Your answer can be represented as a solution of an equation involving trigonometric functions (without trying to solve the equation).

(c) Assume that in the setting of the previous problem two lighthouses are installed on the island and they rotate in the same direction uniformly around the vertical axis, making one revolution per time interval T, and have angle $\theta \in (0, \pi]$ between them. Answer the same question as in the previous item.

Solution Assume for definiteness that the lighthouse rotates counterclockwise.

(a) The answer is **yes**. Call the disc of radius L the *surveillance disc*. Obviously it is the most beneficial for the boat to enter the surveillance disc at the point write after the searchlight beam just passed through it. The trivial strategy for a boat is to sail along the straight line to the island O with the speed $v = 0.9\frac{L}{T}$. The time it requires is equal to $\frac{L}{v} = \frac{10}{9}T > T$ so the boat will be noticed and this strategy does not work. It does not mean that there is no other strategy though.

A more sophisticated strategy is based on the following observation: the linear speed of the searchlight beam's point at a distance of r from O is equal to $\frac{2\pi}{T}r$ and in particular it goes to zero as $r \to 0$. Hence, on the circle of radius

$$r_0 = \frac{vT}{2\pi}. \tag{7}$$

the boat can sail counterclockwise with the beam behind him as its velocity is equal to the linear velocity of the corresponding point on the beam. We will call this circle the *safety circle* (for the boat). Outside this circle the (linear) velocity of the beam is greater than the velocity of the boat and inside of this circle it is smaller.

This implies that if the boat can reach some point A of the safety circle without being noticed then after this it can reach the island O without being noticed. Indeed, one of the possible trajectories inside the safety circle is along the circle from A to O with the diameter OA (see Figure 9):

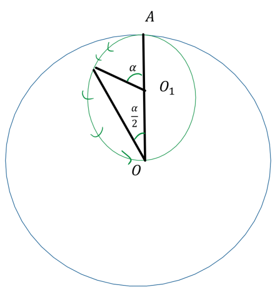

Figure 9

The radius of this smaller circle is $\frac{r_0}{2}$. Assume that O_1 is the center of this smaller circle (i.e. the middle point of OA). If the boat moves along this circle with linear velocity v, then its angular velocity with respect to the center O_1 of the smaller circle is $\omega := \frac{2v}{r_0}$. By the Inscribed Angle Theorem the angular velocity of the ray connecting the island O with the position of the boat on the smaller circles is equal to $\frac{1}{2}\omega = \frac{v}{r_0} = \frac{2\pi}{T}$ (we used (7) in the last equality). i.e. it is equal to the angular velocity of the beam and the boat cannot be reached by the beam.

So, the main goal of the boat is to reach the safety circle. It can be done going along the segment of straight line toward the island O: now the boat needs to sale a smaller distance $L - r_0$ instead of L and the time needed for this is $\frac{L-r_0}{v} = \frac{L}{v} - \frac{T}{2\pi}$. The boat can do it without being noticed if $\frac{L}{v} - \frac{T}{2\pi} < T$, or equivalently

$$v > \frac{1}{1+\frac{1}{2\pi}}\,\frac{L}{T} = \frac{2\pi}{2\pi+1}\,\frac{L}{T}. \tag{8}$$

Note that $\frac{2\pi}{2\pi+1} \sim 0.863 < 0.9$ (in fact, since $3.14 < \pi < 3.15$, $\frac{2\pi}{2\pi+1} = \frac{6.3}{7.28} < \frac{6.3}{7.2} = \frac{7}{8} = 0.875 < 0.9$). Therefore $v = 0.9\frac{L}{T}$ satisfies (8) and the boat can reach the island unnoticed.

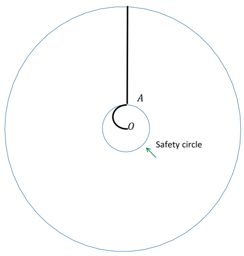

Figure 10

(b) The answer is $\boxed{v = (\cos\beta)\,\frac{2\pi L}{T}}$, where β is the minimal positive solution of the equation

$$2\pi + \beta = \tan\beta. \tag{9}$$

(numerically $v_0 \approx 0.807\frac{L}{T}$). Note that $\frac{2\pi}{T}L$ is the linear velocity of the searchlight beam on the boundary of the disc of surveillance.

An intuitive proof. Denote by S the disc of surveillance and by S_0 the safety circle. Let I be the point of entrance of the boat to S , which is the point the beam just passed (but we assume that the boat was not noticed). WLOG we can assume that it occurs at the time moment $t = 0$ and that I is on the top of the boundary of S (i.e. has the maximal y-coordinate. Denote by B_t the location of the beam at time t and by C_t the set of points in S that can be reached by the boat in time t. C_t is the intersection of the disk of radius vt centered at I with S. Since the beam on the boundary of S is much faster than the boat, at sufficiently small time moments $t > 0$ the beam does not intersect C_t. As the beam rotates it starts to meet the corresponding set C_t. Let t_1 be the first time it occurs. For times $t \geq t_1$ the subset D_t of C_t consisting of all points in C_t that the boat can reach without being noticed is situated from the left of the beam B_t (see Figure 11 below).

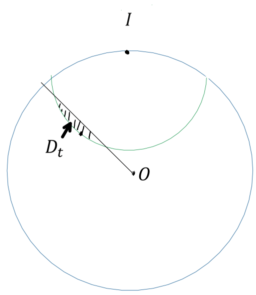

Figure 11

Let t_2 be the time moment such that D_t is not empty for all $t \in [t_1, t_2]$ but is empty when $t > t_2$ (set $t_2 = \infty$ if such time moment does not exist, but the latter case is not interesting as it occurs for sufficiently large velocity v). Geometrically the beam B_{t_2} is tangent to the circle of radius vt_2 centered at I. Denote the point of tangency by E. The safety circle S_0 (and therefore also the island) can be reached by the boat without being noticed if and only if for some $t \in [t_1, t_2)$ the set D_t has nonempty intersection with S_0, which in turns is equivalent to the condition that the point E is inside of S_0. Therefore the threshold occurs for the velocity v_0 for which the point E lies exactly on the safety circle S_0, i.e. if $v > v_0$ the boat can reach the island without being noticed and if $v \geq v_0$ it will be notice (see more rigorous prove of it below).

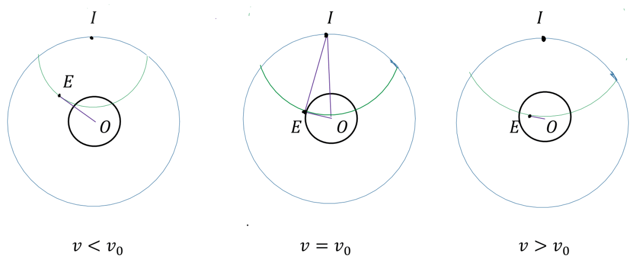

Figure 12

Before going to more rigorous proof, let us find v_0. Let $\beta = \angle EOI$. The total angle the beam made before the time t_2 is $2\pi + \beta$ (with the angular velocity $\frac{2\pi}{T}$). Hence

$$t_2 = \frac{2\pi + \beta}{2\pi} T. \tag{10}$$

Note that $|IO| = L$, $|OE| = r_0 = \frac{v_0 T}{2\pi}$ by (7), $|IE| = v_0 t_2$, and the triangle $\triangle OEI$ is right. Consequently,

$$\tan\beta = \frac{|IE|}{|OE|} = \frac{2\pi t_2}{T}. \tag{11}$$

Plugging (10) into (11) we get the equation (9) for the angle β. Finally, using (11)

$$v_0 = \frac{|EI|}{t_2} = \frac{L\sin\beta}{t_2} = \frac{2\pi L\sin\beta}{T\tan\beta} = \cos\beta\frac{2\pi}{T}L.$$

More rigorous proof. Suppose that the boat can reach the island. Let I be the point where it first enters the surveillance circle. Let E be the point where it first reaches the safety circle. Let IA be the tangent to the safety circle that is located in a counterclockwise direction from IO.

Let us prove that the boat can move along IA instead of IE and still be unnoticed.

First, suppose that E is on the same side of OA as I (see Figure 12 below). Add the arc AE of the safety circle to the trajectory of the boat. The boat stays unnoticed when it moves along this arc, since on the safety circle, its angular speed is equal to the angular speed of the beam.

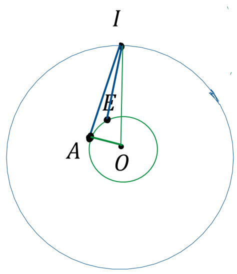

Figure 13

Now, project the trajectory $I - E - A$ of the boat (perpendicularly) onto the line IA. The projection will not increase the speed of motion, so the boat will still be able to proceed along the new trajectory, and will get to A sooner than along its old $I - E - A$ trajectory.

Since the boat stayed unnoticed on the way $I - E - A$, the search beam was behind the boat when it reached A along the old trajectory. Thus the search beam is behind A when the boat reaches A along IA. It is not possible for the search beam to "see" the boat somewhere on its way to A: the angular speed of the beam is greater than that of the boat outside the safety circle, so if the beam is behind the boat at the end of the way, then it was behind the boat all the time. This completes the proof.

If E is on the other side of OA than I (i.e. the boat goes around the safety circle before entering it), the proof is similar, but there is no need to add an arc along the safety circle, since the projection of the trajectory $I-E$ to the line OA contains A. Let A' be the point on the old trajectory $I-E$ that projects to A; then the boat will reach A along IA sooner than it used to reach A' along the old trajectory. Since A' is on the ray OA, this means that the boat will reach A unnoticed.

We conclude that if the boat can reach the safety circle, then it can do this along the line IA.

Let $\alpha = \angle AOI$; since AOI is the right triangle, we have $|IA| = L\sin\alpha$ and $|AO| = L\cos\alpha$; since $|AO| = r_0 = \frac{vT}{2\pi}$, thus $v = \frac{2\pi L}{T}\cos\alpha$.

The beam will reach the point A in time $t = \frac{2\pi+\alpha}{2\pi}T$. For the boat to remain unnoticed, we need to have $vt \geq |IA|$, so that the boat can reach A before the beam reaches it. Thus we have the following condition on α:

$$L\sin\alpha \leq v\frac{2\pi+\alpha}{2\pi}T, \tag{12}$$

and substituting $v = \frac{2\pi L}{T}\cos\alpha$, we get

$$\tan\alpha \leq 2\pi + \alpha$$

i.e. α is smaller than the root $\beta \in (0, \pi/2)$ of the equation $\tan\beta = 2\pi + \beta$. Since cos decreases on $(0, \pi/2)$, we have a lower estimate on v:

$$v = \frac{2\pi L}{T}\cos\alpha \geq \frac{2\pi L}{T}\cos\beta.$$

On the other hand, if v satisfies this inequality (i.e. $v = \frac{2\pi L}{T}\cos\alpha$ for $\tan\alpha \leq 2\pi + \alpha$), then (12) holds, thus the boat can reach the safety circle by moving along IA and the beam will be still behind it. The boat can move as in (a) after it reaches the safety circle. This completes the proof.

(c) The answer is $\boxed{v = \cos\beta\frac{2\pi}{T}L}$, where β is the minimal positive solution of the equation

$$2\pi - \theta + \beta = \tan\beta. \tag{13}$$

One searchlight beam is obtained from the other by the shorter rotation in the counterlockwise. We will call by the *front beam* the former and by *rear beam* the latter (it is ambiguous in the case $\theta = \pi$ as both beams are front and rear in this case but this ambiguity is not important). Obviously, it is beneficial for the boat to enter the surveillance right after the rear beam passes. Then the arguments are exactly the same as in item (b), the only difference is that in the numerator of (10) one has to replace $2\pi + \beta$ by $2\pi - \theta + \beta$, which leads to the equation (13).

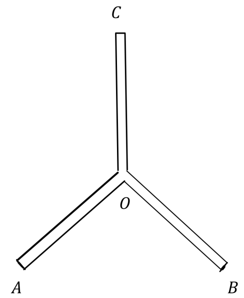

Figure 14

8. Three narrow rectangular hallways of length ℓ meet in the common point as shown in the figure above. One can ignore the width of the hallways so each of them is considered as a line segment. A cop and a gangster run along these hallways so that the maximal speed of the cop is twice of the maximal speed of the gangster. A cop sees the gangster if they are both in the same hallway and the distance between them is not greater than r. This includes the case when one of them is in the center and when the cop is in the center he can watch simultaneously along all three hallways. Prove that under each of the following conditions on r, the cop can catch the gangster from any starting position:

   (a) $r \geq \frac{\ell}{3}$;

   (b) $r \geq \frac{\ell}{4}$;

   (c) $r > \frac{\ell}{5}$;

   (d) $r > \frac{\ell}{7}$.

   Solution. In the sequel we describe the cop winning strategy for all items. In fact, the strategy for every item can be made as the part of the strategy for the next one so the solution of each item is a part of the solution of the next one. In item 8 (c) we give an alternative strategy, which is based on the strategy of item (a) and (d) (skipping the moves in item (b)).

   The Rough Idea *Before going into the details, here is the very rough idea behind the cop's strategy, at least for parts (a)-(c): first the cope combs through one of the hallways (say, OC) in order to make sure the gangster is not there. Then he alternates between visiting the two other hallways OA and OB without going too far down either so that the gangster cannot escape back into the first hallway OC without being noticed. Using the* *advantage in speed, he can dig deeper inside of those hallways upon each visit, repelling the gangster more and more until the gangster will be "pressed against the walls".*

For part (d) this strategy is not sufficient, and once the gangster is repelled enough into one of the hallways OA or OB, the cop combs entirely through one of them , say OA which gives the gangster a possibility to slip through the hallway OC but then it turns to be the wrong move for the gangster, as the cop will be able to catch him. Besides, even the gangster will be clever enough to stay in OB, using sufficiently many alternating visits of OC and OA the cop will know for sure that the gangster is in the hallway OB and finally will catch the gangster by running there.

Note that we also give an alternative and shorter strategy for part (c) but we are not aware if this strategy is applicable to solve part (d). In fact this alternative strategy is the application of our strategy for (d) right after applying the strategy of (a) which is sufficient for the considered case.

Now we start the detailed implementation of this strategy. Assume that the cop always runs with the maximal velocity and starts at the center O.

First, the cop can comb through one of the hallways, say the hallway OC. If the gangster was not noticed, it means that he is not in the hallway OC and is at the distance greater than r from O in the rest of the hallways. Then the cop runs along the hallway OA from the center O for the distance of $2r$ and if he did not see the gangster during this run he comes back to the point O. Note that if during this step the cop still did not see the gangster, then it means that during the same time period the gangster could not move from the hallway OB to the hallway OC without being noticed, because otherwise he need to run the distance greater than $2r$, while the cop made the total distance of $4r$ at the same time period, which contradicts the speed ratio constraints.

(a) If $r \geq \frac{\ell}{3} \Leftrightarrow l \leq 3r$, then the fact that the cop did not see the gangster in the hallway OA means that the gangster is not in this hallway (and also not in the hallway OC by the above), so he is in the hallway OB and the cop will catch him by running there.

(b) Assume that $3r < \ell \leq 4r$. In this case it is possible that the gangster is in the hallway OA, but if this is the case, then at the moment the cop returned to the center O from the hallway OA the gangster is in the distance greater than $2r$ from O.

Now the cop will run into the hallway OB for the distance of $3r$ and returns to O. There are two possibilities:

(a) The gangster was in the hallway OB and will be caught (as $\ell \leq 4r$)

(b) The gangster was in the hallway OA. In this case he has no time to move to the hallway OC without being caught as for this he needs to make the distance greater than $3r$, which is more than half of the distance the cop made during his round-trip to the hallway OB (the cop makes $6r$ during this round-trip). Obviously the gangster will be caught if he will try to move to the hallway OB.

So after returning to O the cop is sure that the gangster is in the hallway OA and he will catch the gangster after combing through it.

(c) Assume that $4r < \ell < 5r$.

Way one This way is shorter, but our solution of the next item (d) is not based on it. In fact this is the application of our strategy of (d), see below, right after applying the strategy of (a) which is sufficient for the considered case.

In more detail, here we describe the strategy after the cop runs the distance $2r$ into the hallway OA and returns to O, as at the end of the strategy in item (a). Then instead of running the distance of $3r$ into the hallway OB, as suggested in item (b), he runs the distance $4r$ there and comes back to O. If he did not see the gangster during this time, it means that the hallway OB is clear. If the gangster moved from OA to OC during this time, then he is in OC no further than the distance of $2r$ (as in the beginning of the cop's visit to OB the gangster was in OA in the distance greater than $2r$ from O). In this case the cop will notice the gangster if he checks OC till $2r$. If the gangster is still not noticed, then he stayed in OA before the last visit of the cop to OC and he has no time to move to OB without being noticed, so the cop is sure that the gangster is in OA and the cop catches him there.

Way two In this case it is possible that the gangster is in the hallway OB, but if this is the case, then at the moment the cop returned to the center O from the hallway OB the gangster is in the distance from O greater than $3r + r - \frac{3r}{2} = \frac{5r}{2}$. The further steps of the cop are to visit alternatively the hallways OA and OB going deeper into this hallways on each further step, but such that the gangster will not be able to move to the hallway OC without being noticed. We will call a *cycle* each such round-trip to the hallways OA and OB. Two such cycles were already described in the solutions to items (a) and (b).

Assume by induction that after the nth cycle the cop is sure that the gangster is not in the hallway OC, and in the hallway OX (where $X = A$ for add n and $X = B$ for even n) the gangster must be in the distance greater than x_n from O. Note that $x_1 = 2r$. Then in the $(n + 1)$st cycle the cop runs to the other hallway (which is not OC) for the distance of $x_n + r$. Then if he did not find the gangster there, he comes back to O, knowing that

(a) the gangster is not in the hallway OC (as the distance the cop made in this cycle is equal to $2(x_n + r)$ and the distance the gangster has to make to stay unnoticed is greater than $x_n + r$, so the gangster cannot make it under the velocity ratio constraints).;

(b) in the hallway OX which was just combed by the cop the gangster is on the distance greater than

$$x_{n+1} := \frac{x_n + r}{2} + r = \frac{x_n + 3r}{2},$$

which implies that $3r - x_{n+1} = \dfrac{1}{2}(3r - x_n)$, i.e. $3r - x_n = 2^{1-n}(3r - x_1) = 2^{1-n}r$, i.e.

$$x_n = 3r - 2^{1-n}r. \tag{14}$$

The gangster will be caught if for some n we have $x_n + 2r \geq \ell$, i.e. $\ell \leq 5r - 2^{1-n}r$. Such n exists as we assumed that $\ell < 5r$.

(d) Assume that $5r \leq \ell < 7r$. Let $\varepsilon = \frac{1}{2}(7r - l)$. The reason for the choice of this ε is the following identity:

$$(\ell - r) - (3r - \varepsilon) = 3r - \varepsilon \tag{15}$$

The cop acts as in part (c) until the step n for which $x_n \geq 3r - \varepsilon$. By (14) such n exists. Then he changes the strategy as follows: He runs to the next hallway in the line (among OA or OB), say OA for definiteness, but now he moves there the distance of $l - r$ and returns to O. If he did not see the gangster along this way, it means the gangster is not in OA (and if the cop sees the gangster than he catches him). During this cycle of the cop, in contrast to the second strategy of part (c), the gangster can run from the hallway OB to hallway OC without being noticed, but if he does it at the moment the cop's last return to O the distance from O to the gangster along the hallway OC is not greater than $\ell - r - (3r - \varepsilon) = 3r - \varepsilon$ from the choice of ε, given by (15).

So, the cop does the following

Elementary move: *If the cop, being at point O, does not see the gangster in the hallway OB (and catch the gangster in this way) the cop runs to the hallway OC for the distance of*

$$l_1 = 4r - 2\varepsilon$$

The reason for this choice of the distance is that l_1 satisfies

$$3r - \varepsilon + \frac{1}{2}l_1 - l_1 = r$$

so that if the gangster is in the hallway OC going the distance l_1 into OC the cop will see the gangster and so will catch him. It means that if the cop did not see the gangster along this move, the gangster is not in OC and the cop returns to O.

During the last cycle of the cop the gangster either stays in OB (keeping the distance greater than r from O) or moves from OB to OA, but in this case at the moment the cop returns to O the distance of him to the gangster is $l_2 - r = 3r - 2\varepsilon$ and we can repeat the elementary move above with OC replaced by OA and ε replaced by 2ε. So that in this way either the gangster is caught or the cope alternates the moves into OA and OC with the distance of going into those hallways equal to $l_n = 4r - 2^n\varepsilon$ (here the elementary move above as the first step). Since l_n becomes negative for sufficiently large n, either the gangster will be caught in one of the hallways OA and OC or the cop is sure the gangster is in the hallway OB and then will catch him there. This complete the proof of item (d).