Math Contest BC Exam Solution
November 4, 2023

Directions: If units are involved, include them in your answer.

1. Suppose the hour hand and minute hand of a clock make an angle of $10°$. Assuming the hours and minutes are integers, determine the first time from midnight to noon that this occurs. Write your answer in the form $h : m$, where h represents hours and m represents minutes.

   Solution. Suppose the clock reads $h : m$ $(0 \le h \le 11,\ 0 \le m \le 59)$. The angle between the hour hand and 12 o'clock is $30h + \frac{30}{60}m$. And the angle between the minute hand and 12 o'clock is $\frac{360}{60}m$. The difference between the two angles must be $10°$, and so we have

   $$30h + \frac{30}{60}m - \frac{360}{60}m = \pm 10°.$$

   Since $60h - 11m = \pm 20$, m is a multiple of 20, or $m = 20s$. Then $h = (11s \pm 1)/3$. Therefore, $11s \pm 1$ must be divisible by 3. The only possibilities respecting the range of h are as follows:

   $$h = \frac{11 \cdot 1 + 1}{3} = 4 \text{ and } h = \frac{11 \cdot 2 - 1}{3} = 7$$

   when $m = 20$ and $m = 40$ respectively. The only integer solutions for h and m are $(h, m) = (4, 20)$ or $(h, m) = (7, 40)$.

   Answer: $h : m = 4 : 20$

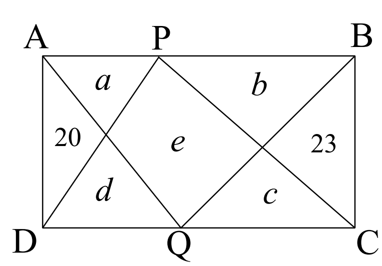
2. Consider the collection of all points, obtained by the reflection of point $B(0, 1)$ across every line passing through point $A(1, 0)$. What is the length of the curve formed by connecting these reflected points?

   Solution. Since $PA = BA$, the collection of points P forms a circle with radius $\overline{AB}$ centered at A. Indeed, every point P on the circle is the reflection of B across the line that passes through point A and bisects angle BAP. The length of the circle is $2\pi\sqrt{2}$.

   Answer. $2\pi\sqrt{2}$

3. Given rectangle $ABCD$, points P and Q lie on sides $\overline{AB}$ and $\overline{CD}$ respectively. The line segments $\overline{PC}$, $\overline{PD}$, $\overline{QA}$, and $\overline{QB}$ collectively form six triangles and one quadrilateral. Determine the area of this quadrilateral when the areas of the two triangles containing sides $\overline{AD}$ and $\overline{BC}$ are given as 20 and 23 respectively.

   Solution.



Let $a, b, c, d,$ and e denote areas of remaining triangles and the quadrilateral as in the figure. The area of $\triangle PDC$ equals the sum of the areas of $\triangle ADQ$ and $\triangle BQC$ because both are equal to half of the area of the rectangle. Consequently,

$$d + e + c = (20 + d) + (c + 23) \quad \Rightarrow \quad e = 43.$$

Answer. 43

4. Suppose the sum of the lengths of all edges of a rectangular prism (or a cuboid) is 64, and the length of a diagonal is $7\sqrt{2}$. Find the surface area of the rectangular prism.

   Solution. Let a, b, and c denote the edges of the rectangular prism. From the condition, we have

$$4(a + b + c) = 64 \quad \text{or} \quad a + b + c = 16,$$

   and

$$a^2 + b^2 + c^2 = (7\sqrt{2})^2 = 98.$$

   Consequently, the surface area is

$$2(ab + bc + ca) = (a + b + c)^2 - a^2 + b^2 + c^2 = 16^2 - 98 = 158.$$

   Answer. 158

5. How many natural numbers less than or equal to 1000 have exactly 3 factors?

   Solution. If a natural number n has two or more distinct prime factors, then it will have 4 or more factors. Therefore, natural numbers with exactly 3 factors have only one prime factor. Consequently, natural numbers with precisely 3 factors must be of the form $n = p^2$ for some prime p. From the computation $31^2 = 961$ and $32^2 = 1024$, there are 11 natural numbers

$$2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, \text{ and } 31^2.$$

   Answer. 11

6. What is the value of $\sqrt{11 \cdot 12 \cdot 13 \cdot 14 + 1}$?

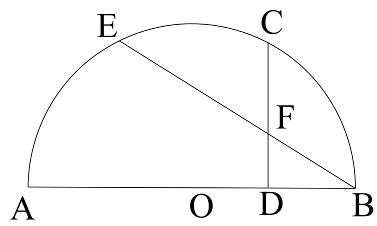
   Solution. We can simplify the expression by rewriting it as follows.

$$11 \cdot 12 \cdot 13 \cdot 14 + 1 = (11 \cdot 14)(13 \cdot 12) + 1 = 154 \cdot 156 + 1 = (155 - 1)(155 + 1) + 1 = 155^2,$$

   Therefore, the given expression is indeed equal to 155.

   Answer. 155

7. Suppose C and E are on the semicircle with diameter $AB = 3$. Let D be a point on the segment $\overline{AB}$ such that $\overline{CD} \perp \overline{AB}$ and F is the point of intersection of $\overline{EB}$ and $\overline{CD}$ as in the figure. Find $\dfrac{BE}{BD}$ if $BF = 1$.



Solution. Draw $\overline{AE}$ to see that the four points A, E, F, and D are on a circle with the diameter $\overline{AF}$. The secant lines of this circle satisfy

$$\overline{BE} \cdot \overline{BF} = \overline{BD} \cdot \overline{BA} \quad \Rightarrow \quad \overline{BE} = \overline{BD} \cdot 3 \quad \Rightarrow \quad \frac{BE}{BD} = 3.$$

Answer. 3

8. Find $x^6 + y^6$ if $x + y = 1$ and $x^3 + y^3 = 16$.

Solution. From the condition, we have

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y) = 16 + 3xy \quad \Rightarrow \quad 1 = 16 + 3xy \quad \text{or} \quad xy = -5$$

and

$$x^6 + y^6 = (x^3 + y^3)^2 - 2x^3y^3 = 16^2 - 2(-5)^3 = 256 + 250 = 506.$$

Answer. 506

9. Consider a sequence of numbers $1000^2, 1001^2, 1003^2, \ldots$. Erase the two last digits from each of these numbers. How many first terms in the resulting sequence form an arithmetic progression?

Solution. Let the initial sequence be $a_n = (1000+n)^2$, $n = 0, 1, \ldots$. Then

$$a_n = 10^6 + 2000n + n^2$$

Hence the resulting sequence is $b_n = \lfloor \frac{a_n}{100} \rfloor = 10^4 + 20n + \lfloor \frac{n^2}{100} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer which is not greater than x. Hence the difference between consecutive terms of the sequence will be equal to 20 until $\lfloor \frac{n^2}{100} \rfloor = 0$, i.e., $n^2 < 100$. Since n is nonnegative integer, we get that $0 \leq n \leq 9$, so the first 10 terms of the sequence form the arithmetic progression.

Answer. 10

10. How many three-digit numbers satisfy the following property: two of their digits are equal, and the third one differs from these by 1?

Solution. If the two digits are equal to 0 and the third one is 1, there is one such number, i.e., 100. If the two digits are $1, \dots, 8$ and the third one is bigger by 1, we have $8 \times 3 = 24$ such numbers. If two digits are equal to $2, \dots, 9$ and the third one is smaller, we have $8 \times 3 = 24$ such numbers. Finally, if two digits are 1 and the third one is zero, this is 101 or 110.

Answer. 51

11. Suppose that increasing the bus fare by $x\%$ results in a $\frac{x}{2}\%$ decrease in the number of passengers. To achieve an 8% increase in revenue, what percentage increase in the fare should be implemented? Find the required fare increase percentage assuming that the fare increase does not exceed 50%.

Solution. Let a and b denote the original cost and the number of passengers respectively. The cost and the number of passengers after the increase are

$$a\left(1+\frac{x}{100}\right) \quad \text{and} \quad b\left(1-\frac{x}{200}\right).$$

respectively. The revenue becomes

$$ab\left(1+\frac{x}{100}\right)\left(1-\frac{x}{200}\right) = ab\left(1+\frac{8}{100}\right) \quad \Rightarrow \quad \frac{x}{200} - \frac{x^2}{20000} = \frac{8}{100}.$$

Solving $x^2 - 100x + 1600 = 0$, we have $(x-20)(x-80) = 0$ or $x = 20$ since $x \le 50$.

Answer. 20%

12. What is the value of the natural number n for which the number of factors of $2^n(3^n + 3^{n+1})$ is 99?

Solution. By rewriting

$$2^n(3^n + 3^{n+1}) = 2^n 3^n (1+3) = 2^{n+2}3^n,$$

we see that the number of factors is

$$(n+3)(n+1) = 99 \quad \text{or} \quad n^2 + 4n - 96 = (n-8)(n+12) = 0.$$

So $n = 8$

Answer. 8

13. In $BC + EXAM = 2023$, all letters correspond to different digits, $B \neq 0$, $E \neq 0$. Among all solutions, find the maximal possible value of $EXAM$.

Solution. Since $EXAM < 2023$ and $E \neq 0$, we have $E = 2$ or $E = 1$. If $E = 2$, then $X = 0$, $BC + AM = 23$. Note that neither of B nor A can be 0 or 2, hence $A = B = 1$ which is impossible. Thus $E = 1$. The largest number that starts with 1 and consists of four distinct digits is 1987. If $EXAM = 1987$, then $BC = 36$.

Answer. 1987

14. Consider a function $f : X \to X$ for the set X of non-negative integers. Find $f(2023)$ if $f(f(n)) + 2f(n) = 3n + 4$.

Solution. For $n = 0$, we have $2f(0) = 4 - f(f(0)) \leq 4$ since $f(f(n)) \geq 0$. We consider cases $f(0) = 0, 1, 2$ and show that $f(0) = 1$.

If $f(0) = 0$, we have

$$f(f(0)) + 2f(0) = 3 \cdot 0 + 4 \quad \Rightarrow \quad f(0) = 4,$$

which is contradictory. If $f(0) = 2$, we have

$$f(f(0)) + 2f(0) = 3 \cdot 0 + 4 \quad \Rightarrow \quad f(2) + 4 = 4 \quad \text{or} \quad f(2) = 0.$$

However, this would imply, for $n = 2$,

$$f(f(2)) + 2f(2) = 3 \cdot 2 + 4 \quad \Rightarrow \quad f(0) = 10,$$

which is contradictory.

From $f(0) = 1$, we have

$$f(f(0)) + 2f(0) = 4 \quad \Rightarrow \quad f(1) = 2.$$

Inductively, one can show that $f(n) = n + 1$ for all $n \in X$. If we assume that $f(k) = k + 1$, then

$$f(f(k))+2f(k) = 3k+4 \quad \Rightarrow \quad f(k+1)+2(k+1) = 3k+4 \quad \Rightarrow \quad f(k+1) = 3k+4-2(k+1) = k+2$$

for any $k \in X$. Therefore $f(2023) = 2024$.

Answer. 2024

15. If a positive integer n can be represented as three-digit numbers, abc in base 6 and cab in base 9, what is the decimal representation of n?

Solution When expressed in base 6 and 9,

$$abc_6 = 36a + 6b + c \quad \text{and} \quad cab_9 = 81c + 9a + b$$

Simplifying $36a + 6b + c = 81c + 9a + b$, we have

$$27a + 5b - 80c = 0 \quad \Rightarrow \quad 27a = 5(16c - b),$$

which implies $a = 5$ and $16c - b = 27$. Since $0 \leq b \leq 9$,

$$27 \leq 16c = b + 27 \leq 36.$$

Thus $c = 2$ and $b = 5$. Now $n = abc_6 = 36 \cdot 5 + 6 \cdot 5 + 2 = 212$.

Answer. 212

16. There are 15 1's arranged in a row, and you can insert either a plus (+) or a minus (-) sign between every two consecutive 1's. How many different ways can you do this such that the result of the calculation equals 7?

Solution. First, there are 14 spaces between consecutive 1's where a plus (+) or minus (-) sign can be placed. Now, if we assume that a minus (-) sign is placed in x out of these 14 spaces, while the remaining $14 - x$ spaces have plus (+) signs, then the calculation results in

$$1 - x + (14 - x) = 15 - 2x,$$

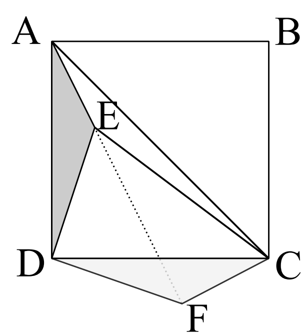
which becomes 7 when $x = 4$. Therefore, the number of ways to place plus (+) and minus (-) signs to make the calculation result equal to 7 is as follows.

$$\binom{14}{4} = \frac{14 \times 13 \times 12 \times 11}{4 \times 3 \times 2 \times 1} = 1001.$$

Answer. 1001

17. Suppose E is a point inside the square $ABCD$ with $AE = 1$, $DE = 2$, and $CE = 3$. Find the diagonal AC.

Solution. Rotate $\triangle ADE$ by $90°$ about D to have $\triangle CDF$ as in the figure. Note that $\angle EDF = 90°$ and $ED = FD = 2$, and so $\angle DEF = \angle DFE = 45°$.



On $\triangle EFC$, $EF = 2\sqrt{2}$, $EC = 3$, and $CF = 1$, which implies $\angle EFC = 90°$ by the Pythagorean theorem. We can also show E is on $\overline{AF}$ since $\angle AEF = \angle AED + \angle DEF = 180°$ which follows from $\angle AED = \angle CFD = 90° + 45°$. Now, on the right triangle $\triangle AFC$, we have

$$AC = \sqrt{(1 + 2\sqrt{2})^2 + 1^2} = \sqrt{10 + 4\sqrt{2}}.$$

Answer. $\sqrt{10 + 4\sqrt{2}}$

18. Let A and B be two objects initially positioned at opposite points along a straight line. When they both travel at their original constant speeds, it takes 30 minutes for them to meet each other. If A doubles their speed while B maintains the original speed, they meet in 25 minutes. Determine the time in minutes it will take for them to meet if B doubles their speed while A retains the original speed.

Solution. Let $AB = \ell$, and s_1 and s_2 denote the original speed of A and B respectively. From the condition, we have

$$(s_1 + s_2)30 = \ell \quad \text{and} \quad (2s_1 + s_2)25 = \ell$$

or

$$s_1 + s_2 = \frac{\ell}{30} \quad \text{and} \quad 2s_1 + s_2 = \frac{\ell}{25}.$$

Eliminating s_1 and s_2, we have

$$s_1 = \ell\left(\frac{1}{25} - \frac{1}{30}\right) \quad \text{and} \quad s_2 = \ell\left(\frac{2}{30} - \frac{1}{25}\right)$$

We want to find t such that

$$(s_1 + 2s_2)t = \ell.$$

From

$$s_1 + 2s_2 = \ell\left(\frac{1}{25} - \frac{1}{30}\right) + 2\ell\left(\frac{2}{30} - \frac{1}{25}\right) = \ell\left(\frac{1}{25} - \frac{1}{30} + \frac{4}{30} - \frac{2}{25}\right) = \ell\left(\frac{3}{30} - \frac{1}{25}\right) = \ell\frac{6}{100}$$

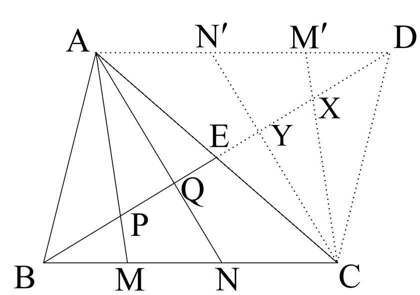
we have $t = 100/6$ min because

$$(s_1 + 2s_2)\frac{100}{6} = \ell$$

Answer. $\frac{50}{3}$ min or $16\frac{2}{3}$ min

19. Given triangle $\triangle ABC$, suppose M and N are trisection points of $\overline{BC}$, and $\overline{BE}$ is a median. Line segments $\overline{AM}$ and $\overline{AN}$ divide $\overline{BE}$ into three parts with ratios $a : b : c$. Find the ratio $a : b : c$.

Solution. Consider the reflection of $\triangle ABC$ about E as in the figure below. We have $\overline{AM} \parallel \overline{CM'}$ and $\overline{AN} \parallel \overline{CN'}$.



From the similarity $\triangle APD \sim \triangle M'XD$ with the ratio $AD/M'D = 3/1$,

$$\frac{AM'}{M'D} = \frac{2(b+c)}{a} = \frac{2}{1} \quad \Rightarrow \quad a = b + c.$$

Similarly, from the similarity $\triangle AQD \sim \triangle N'YD$ with the ratio $AD/N'D = 3/2$,

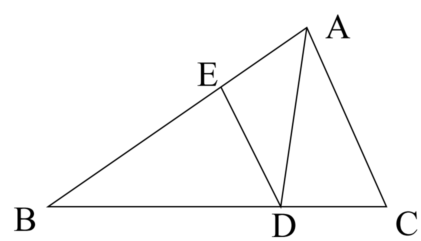
$$\frac{N'D}{AN'} = \frac{a+b}{2c} = \frac{2}{1} \quad \Rightarrow \quad a + b = 4c.$$

Thus we have

$$a : b : c = \frac{5}{2}c : \frac{3}{2}c : c = 5 : 3 : 2.$$

Answer: $5 : 3 : 2$

20. Suppose $\angle EBD = \angle EDA = \angle DAC$ for points E and D in $\triangle ABC$. Let m_1, m_2, and m_3 be perimeters of $\triangle ABC$, $\triangle EBD$, and $\triangle ADC$ respectively. Find the maximum of $\dfrac{m_2 + m_3}{m_1}$.



Solution. Let $BC = a$, $AC = b$, and $AB = c$. From the condition on congruent angles, we have

$$\overline{ED} \parallel \overline{AC} \quad \text{and} \quad \angle ACD = \angle EDB,$$

which imply similarities $\triangle ABC \sim \triangle EBD \sim \triangle DAC$. From $\triangle DAC \sim \triangle ABC$ and $AC = b$,

$$\frac{DC}{b} = \frac{AD}{c} = \frac{AC}{a} = \frac{b}{a}$$

So $m_3 = DC + AD + AC = (b + c + a)\dfrac{b}{a}$ or

$$\frac{m_3}{m_1} = \frac{b}{a}$$

since $m_1 = a + b + c$.

Since $\triangle EBD \sim \triangle ABC$ and $BD = a - DC = \dfrac{a^2 - b^2}{a}$,

$$\frac{m_2}{m_1} = \frac{BD}{BC} = \frac{a^2 - b^2}{a^2}$$

Now

$$\frac{m_2 + m_3}{m_1} = \frac{a^2 - b^2}{a^2} + \frac{b}{a} = 1 - \frac{b^2}{a^2} + \frac{b}{a} = -\left(\frac{b}{a} - \frac{1}{2}\right)^2 + \frac{5}{4} \le \frac{5}{4}$$

Answer. $\dfrac{5}{4}$