DE Exam Solutions
Texas A&M High School Math Contest
November, 2023

1. Last year John received an inheritance. He payed 20% in federal taxes on the inheritance, and payed 10% of what he had left in state taxes. He payed a total of $10,500 for both taxes. How many dollars was the inheritance?

If x is the inheritance, then the federal inheritance tax was $x/5$, and $4x/5$ was left. So, he payed $4x/50 = 2x/25$ of state tax. Together, he payed $x/5 + 2x/25 = 7x/25$ in taxes. So, $7x/25 = \$10,500$, hence $x = \$37,500$.

Answer: $\$37,500$.

2. The first term of a sequence is 2023. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2023rd term of the sequence?

The sequence is 2023, $8+8+27 = 43$, $64+27 = 91$, $729+1 = 730$, $343+27 = 370$, $27+343 = 370$, and we see that the sequence is constant starting from the fifth term.

Answer: 370

3. Two positive numbers are such that their difference, sum, and product are to one another as $1 : 7 : 24$. Find their sum.

Let x, y be these numbers, where $x > y$. We have then $x+y = 7(x-y)$, and $xy = 24(x-y)$. We get $6x = 8y$ from the first equation, hence $y = \frac{3x}{4}$. Substituting into the second equation, we get $\frac{3x^2}{4} = 6x$, hence $x^2 = 8x$. Since $x = 0$ implies $y = 0$, which is not allowed, we get $x = 8$, hence $y = 6$. So, their sum is $x + y = 14$.

Answer: 14.

4. The difference of the sum of legs of a right triangle and its hypothenuse is 8 inches. The height of the triangle drawn to the hypotenuse is 9.6 inches. Find the length of the longer leg.

Let a, b, c be the lengths of the sides of the triangle, where c is the length of the hypothenuse. The area of the triangle is then $\frac{ab}{2}$, so the height is $\frac{ab}{c}$. We have then the system of equations

$$\begin{cases} a^2+b^2 &= c^2 \\ a+b &= 8+c \\ \frac{ab}{c} &= 9.6 \end{cases}$$

Square of the second equation is $a^2 + 2ab + b^2 = 64 + 16c + c^2$. We get then, using the first equation, $c^2 + 2ab = 64 + 16c + c^2$, hence $ab = 32 + 8c$. Using the third equation, we get $9.6c = 32 + 8c$, hence $1.6c = 32$, so $c = 20$. It follows that $a + b = 28$ and $ab = 192$. Consequently, a and b are the roots of the quadratic polynomial $x^2 - 28x + 192$. It follows that they are equal to $\frac{28\pm\sqrt{28^2-4\cdot 192}}{2} = \frac{28\pm 4}{2}$. Consequently, the legs of the triangle are 16 and 12.

Answer: 16 in.

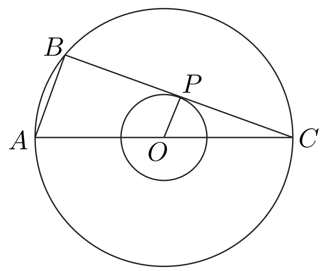
5. A regular polygon with n sides is inscribed in a circle of radius R. What is n, if the area of the polygon is $3R^2$?

Drawing the radii from the center to the vertices of the polygon, we get n equilateral triangles. Area of each of them is $\frac{R^2 \sin \frac{2\pi}{n}}{2}$. Consequently, the area of the polygon is $\frac{nR^2 \sin \frac{2\pi}{n}}{2} = 3R^2$. It follows that $\sin \frac{2\pi}{n} = \frac{6}{n}$. We get the equation $\sin x = \frac{6x}{2\pi}$ for $x = \frac{2\pi}{n}$. Since $x = 0$ is one solution, and $\sin x$ is concave on $(0, \pi)$, there is at most one more solution in this interval. Since $\sin \frac{\pi}{6} = \frac{1}{2}$, $x = \pi/6$, i.e., $n = 12$ is the only solution.

Answer: $n = 12$.

6. The ratio of the radii of two concentric circles is 1:3. Let $\overline{AC}$ be a diameter of the larger circle, let $\overline{BC}$ be a chord of the larger circle that is tangent to the smaller circle. Find the radius of the larger circle if $AB = 12$.

Let O be the center of the circles, and let P be the tangency point of BC and the smaller circle. Then $\angle OPC$ is right. Since $\overline{AC}$ is a diameter of the larger circle, and B is on the larger circle, $\angle ABC$ is also right. Consequently, $\triangle ABC$ and $\triangle OPC$ are similar. Since $OP : OC = 1 : 3$, we have $AB : AC = 1 : 3$, hence $AC = 36$.



Answer: 18.

7. Find all roots of the equation $\sqrt{1 + \sqrt{1 + x}} = x$.

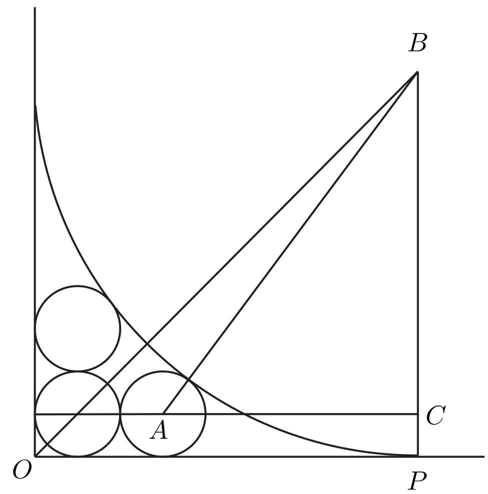
Squaring the equation, we get $1 + \sqrt{1 + x} = x^2$. This implies that $1 + x = (x^2 - 1)^2$, i.e., $1 = (x - 1)^2(x + 1)$, since $x = -1$ is not a root. We have $1 = (x^2 - 2x + 1)(x + 1)$, so that $1 = x^3 - x^2 - x + 1$, hence $x^3 - x^2 - x = 0$. Since $x = 0$ is also not a root, we get $x^2 - x - 1 = 0$, hence $x = \frac{1 \pm \sqrt{5}}{2}$. The root $\frac{1 - \sqrt{5}}{2}$ of the quadratic equation not a solution of our original equations, since it is negative.

The only remaining case is $x_1 = \frac{1+\sqrt{5}}{2}$. Let us check that it is a solution. We have $1 + x_1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$, so $x_1 = \sqrt{1 + x_1}$. Consequently, $x_1 = \sqrt{1 + \sqrt{1 + x_1}}$.

Answer: $\frac{1+\sqrt{5}}{2}$.

8. Three circles of radius s are drawn inside a right angle. The first circle is tangent to both sides of the angle, the second is tangent to the first circle and one side of the angle, and the third is tangent to the first circle and the other side of the angle. A circle of radius $r > s$ is tangent to both sides of the angle and to the second and the third circles. Find r/s.

Let A be the center of the second circle, and let B be the center of the circle of radius r. Let AC and BC be lines parallel to the sides of the angle, where AC is parallel to the side that is tangent to the second circle, see the figure. Let O be the vertex of the right angle, and let P be the intersection of the line $\overline{BC}$ with the side of the angle, parallel to $\overline{AC}$, see the figure.

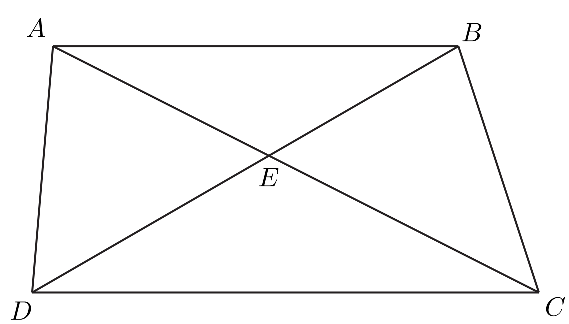


The $\triangle BOP$ is an isosceles right triangle. Consequently, $OP = r$. It follows that $AC = r - 3s$. The triangle $\triangle ABC$ is right. Its legs are $AC = r - 3s$ and $BC = r - s$, and its hypothenuse is $AB = r + s$. It follows that $(r-3s)^2 + (r-s)^2 = (r+s)^2$, i.e., $r^2 - 6rs + 9s^2 + r^2 - 2rs + s^2 = r^2 + 2rs + s^2$. Consequently, $r^2 - 10rs + 9s^2 = 0$. Dividing it by s^2, we get $\left(\frac{r}{s}\right)^2 - 10\frac{r}{s} + 9 = 0$. Consequently, $\frac{r}{s} = \frac{10 \pm \sqrt{100-36}}{2} = \frac{10 \pm 8}{2}$, i.e., $\frac{r}{s} \in \{9, 1\}$. Since $r > s$, we get $\frac{r}{s} = 9$.

Answer: $\frac{r}{s} = 9$.

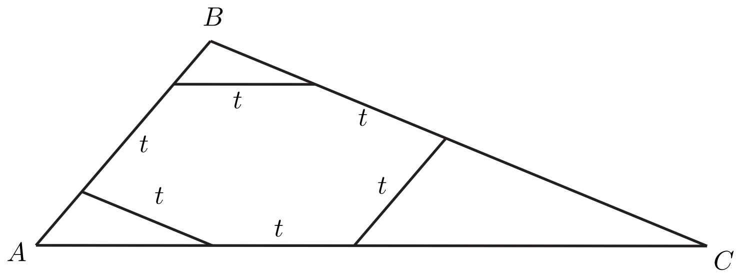
9. Trapezoid is divided by its diagonals into four parts. What is the area of the trapezoid if the areas of the parts adjacent to the bases are 16 and 25?

Let $ABCD$ be our trapezoid, where AB and CD are bases. Let E be the intersection point of the diagonals AC and BD. We assume that areas of $\triangle ABE$ and $\triangle CDE$ are 16 and 25, respectively. Since $AB \parallel CD$, the triangles $\triangle ABE$ and $\triangle CDE$ are similar with the ratio $4:5$. In particular, $BE : ED = 4:5$. This implies that area of $\triangle AEB$ is to area of $\triangle AED$ as $4:5$. Consequently, area of $\triangle AED$ is 20. By the same argument, area of $\triangle BEC$ is 20. Consequently, the area of the trapezoid is $16 + 25 + 20 + 20 = 81$.



Answer: 81.

10. Three lines parallel to the sides of a triangle cut from it three smaller triangles, so that a hexagon remains. If all sides of the hexagon are of the same length, and the lengths of the sides of the triangle are a, b, c, then what is the length of a side of the hexagon?



Let A, B, C be the vertices of the triangle, so that $a = BC, b = AC, c = AB$. The smaller triangles are similar to the original one. Let t be the length of the side of a hexagon. Then the sides of the triangle at the vertex A are $t, \frac{bt}{a}, \frac{ct}{a}$. The sides of the triangle at the vertex B are $\frac{at}{b}, t, \frac{ct}{b}$, and the sides of the triangle at the vertex C are $\frac{at}{c}, \frac{bt}{c}, t$. We get $t = a - \frac{at}{b} - \frac{at}{c}$, hence $\frac{t}{a} = 1 - \frac{t}{b} - \frac{t}{c}$, which implies that $t = \frac{1}{1/a+1/b+1/c}$.

Answer: $\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$.

11. Six distinct integers are picked at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

The second smallest is 3 if and only if 1 and 3 or 2 and 3 are chosen from $\{1, 2, 3\}$. The remaining 4 integers are chosen from $\{4, 5, \ldots, 10\}$. It follows that the probability is equal to the double of the number of ways to choose 4 integers out of 7 divided by the number of ways to choose 6 integers out of 10, i.e., to $\frac{2\binom{7}{4}}{\binom{10}{6}} = \frac{2\cdot 7!6!4!}{4!3!10!} = \frac{2\cdot 4\cdot 5\cdot 6}{8\cdot 9\cdot 10} = \frac{1}{3}$.

Answer: 1/3.

12. A sphere is inscribed in a tetrahedron $ABCD$. The four planes tangent to the sphere parallel to (but distinct from) the planes of the four faces of $ABCD$ cut off smaller tetrahedra. Let F_{ABCD} be the sum of length of the edges of the four smaller tetrahedra divided by the sum of the lengths of the edges of $ABCD$. What is the largest possible value of F_{ABCD} among all tetrahedra $ABCD$.

The four tetrahedra are similar to the tetrahedron $ABCD$. Let k_1, k_2, k_3, k_4 be the proportionality coefficients (less than 1). Then we are asked to find $k_1 + k_2 + k_3 + k_4$. Let r be the radius of the sphere, and let S_1, S_2, S_3, S_4 be the areas of the faces of the tetrahedron $ABCD$, where the face of area S_i is parallel to the plane cutting off the smaller tetrahedron whose proportionality coefficient is k_i. Let V be the volume of the tetrahedron. Then the height of the tetrahedron to the face with area S_i is equal to $3V/S_i$. The height of the smaller tetrahedron cut off by the plane parallel to the face of area S_i is then $\frac{3V}{S_i} - 2r$. It follows that $k_i = \frac{\frac{3V}{S_i} - 2r}{3V/S_i} = \frac{3V - 2rS_i}{3V}$.

The segments from the center of the sphere to the vertices A, B, C, D together with the edges of $ABCD$ form four tetrahedra each sharing a face with $ABCD$. Their heights are radii of the sphere, so their volumes are $S_i r/3$. It follows that $V = \frac{1}{3}(S_1 + S_2 + S_3 + S_4)r$. It follows that

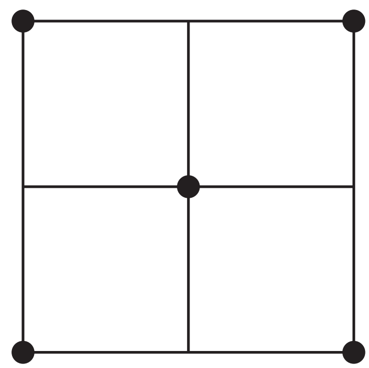
$$k_1 + k_2 + k_3 + k_4 = \frac{12V - 2r(S_1 + S_2 + S_3 + S_4)}{3V} = \frac{12V - 6V}{3V} = 2.$$

In particular it does not depend on the tetrahedron.

Answer: 2.

13. Five points are chosen inside or on a square of side 1. Find the smallest possible a such that it is always possible to select one pair of points from these five such that the distance between them is equal to or less than a.

Let us divide the square into four squares half its size. If we have five points in the square, two of them will be inside one of the squares (or on its boundary). But any two points of a square with side $1/2$ are on distance at most $\sqrt{2}/2$ from each other. It follows that we can always take $a = \sqrt{2}/2$.



Choosing the vertices and the center of the square we get 5 points for which we can not choose a less than $\sqrt{2}/2$.

Answer: $\sqrt{2}/2$.

14. Determine the number of real solutions of the equation

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{3} \right\rfloor + \left\lfloor \frac{a}{5} \right\rfloor = a,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Note that a is an integer. Write $a = 30b + r$, where b is an integer, and $r \in [0, 30)$. (In other words, take $b = \lfloor \frac{a}{30} \rfloor$.) Then $\lfloor \frac{a}{2} \rfloor + \lfloor \frac{a}{3} \rfloor + \lfloor \frac{a}{5} \rfloor = \lfloor 15b + \frac{r}{2} \rfloor + \lfloor 10b + \frac{r}{3} \rfloor + \lfloor 6b + \frac{r}{5} \rfloor = 31b + \lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{3} \rfloor + \lfloor \frac{r}{5} \rfloor$. Consequently, the equation is satisfied if and only if $b = r - \left(\lfloor \frac{r}{2} \rfloor + \lfloor \frac{r}{3} \rfloor + \lfloor \frac{r}{5} \rfloor \right)$. As r is an arbitrary number from the set $\{0, 1, 2, \ldots, 29\}$, we conclude that the equation has 30 solutions.

Answer: 30.

15. Let $f(x)$ and $g(x)$ be different quadratic polynomials of the form $x^2 + px + q$ such that $f(1) + f(10) + f(100) = g(1) + g(10) + g(100)$. Find all x such that $f(x) = g(x)$.

Let $f(x) = x^2 + p_1 x + q_1$ and $g(x) = x^2 + p_2 x + q_2$. Then $1 + p_1 + q_1 + 100 + 10p_1 + q_1 + 10,000 + 100p_1 + q_1 = 1 + p_2 + q_2 + 100 + 10p_2 + q_2 + 10,000 + 100p_2 + q_2$. It follows that $111p_1 + 3q_1 = 111p_2 + 3q_2$, hence $37(p_1 - p_2) = (q_2 - q_1)$. If $p_1 = p_2$, then we get $q_1 = q_2$, but this contradicts $f(x) \neq g(x)$. Consequently, $37 = \frac{q_2 - q_1}{p_1 - p_2}$.

If $x^2 + p_1 x + q_1 = x^2 + p_2 x + q_2$, then $(p_1 - p_2)x = q_2 - q_1$, which implies $x = 37$.

16. The function $F(x)$ is defined for all real numbers except 0 and 1, and satisfies

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$$

for all $x \neq 0, 1$. Find $F(5)$.

Substituting $x = 5$, we get $F(5) + F(4/5) = 6$. Substituting $x = 4/5$, we get $F(4/5) + F(-1/4) = 9/5$. Substituting $x = -1/4$, we get $F(-1/4) + F(5) = 3/4$. We get a system

$$\begin{cases} F(5) + F(4/5) &= 6 \\ F(4/5) + F(-1/4) &= 9/5 \\ F(-1/4) + F(5) &= 3/4 \end{cases}$$

Adding them together, we get $2(F(5) + F(4/5) + F(-1/4)) = \frac{171}{20}$, hence $F(5) + F(4/5) + F(-1/4) = \frac{171}{40}$. Subtracting from it the second equation of the system, we get

$$F(5) = \frac{99}{40}$$

Answer: $\frac{99}{40} = 2\frac{19}{40} = 2.475$

17. Find the largest prime p for which $p^2 - 2$, $2p^2 - 1$, and $3p^2 + 4$ are prime numbers.

A square of an integer can have only residues 0, 1, 2, 4 when divided by 7. If p^2 is divisible by 7, then $p = 7$. We have then $p^2 - 2 = 47$, $2p^2 - 1 = 97$, and $3p^2 + 4 = 151$, which are primes.

If p^2 is 1 modulo 7, then $3p^2 + 4$ is divisible by 7, hence $3p^2 + 4 = 7$. But this implies $p = 1$, which is not allowed.

If p^2 is 2 modulo 7, then $p^2 - 2 = 7$, hence $p = 3 < 7$. We have then $p^2 - 2 = 7$, $2p^2 - 1 = 17$, $3p^2 + 4 = 31$, which are primes.

If p^2 is 4 modulo 7, then $2p^2 - 1 = 7$, hence $p = 2 < 7$. (Then $3p^2 + 4 = 16$, which is not a prime.)

Answer: 7.

18. How many pairs (x, y) of positive integers exist such that $\sqrt{x} + \sqrt{y} = \sqrt{2023}$?

Since $2023 = 7 \cdot 17^2$, the equation is equivalent to $\sqrt{7x} + \sqrt{7y} = 7 \cdot 17$. Suppose (x, y) is a solution, where x and y are positive integers. Then $7y = (7 \cdot 17 - \sqrt{7x})^2$, which implies that $2 \cdot 7 \cdot 17 \cdot \sqrt{7x} = (7 \cdot 17)^2 + 7x - 7y$. We obtain that $\sqrt{7x}$ is a rational number. Since its square $7x$ is an integer, it follows that $\sqrt{7x}$ is, in fact, an integer. Similarly, we obtain that $\sqrt{7y}$ is an integer as well. Let $a = \sqrt{7x}, b = \sqrt{7y}$. Since the integers $a^2 = 7x$ and $b^2 = 7y$ are divisible by 7, so are the integers a and b. Let $a_1 = a/7$ and $b_1 = b/7$. Then $x = 7a_1^2$, $y = 7b_1^2$, and the equation becomes $a_1\sqrt{7} + b_1\sqrt{7} = 17\sqrt{7}$, i.e., $a_1 + b_1 = 17$. There are 16 pairs of positive integers (a_1, b_1) such that $a_1 + b_1 = 17$, so there are 16 pairs of positive integers (x, y) such that $\sqrt{x} + \sqrt{y} = \sqrt{2023}$.

Answer: 16.

19. Let each of the numbers $x_1, x_2, \ldots, x_{2023}$ be independently equal to -1, 0, or 1. What is the smallest possible value of the sum of their pairwise products $\sum_{1 \leq i < j \leq 2023} x_i x_j$?

The sum is equal to $\frac{1}{2}\left((x_1+x_2+\cdots+x_{2023})^2-(x_1^2+x_2^2+\cdots+x_{2023}^2)\right)$.Let a and b count how many of the numbers $x_1, x_2, \ldots, x_{2023}$ equal 1 and -1, respectively. Then the sum of the pairwise products is equal to $\frac{1}{2}((a-b)^2-(a+b))$.

The largest possible value of $a+b$ is 2023. If $a+b=2023$ then $a \neq b$ so that $(a-b)^2 \geq 1$. Hence the sum is at least $(1-2023)/2=-1011$. The value -1011 is attained when $a=1012$, $b=1011$ (or $a=1011$, $b=1012$). In the case $a+b<2023$, the sum is at least $(0-2022)/2=-1011$. The value -1011 is attained when $a=b=1011$. Otherwise the sum is strictly larger.

Answer: -1011.

20. Call a set of integers *sparse* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1,2,\ldots,12\}$, including the empty set, are sparse?

Let S_n be the number of sparse subsets of $\{1,2,\ldots,n\}$. We have $S_1=2$, $S_2=3$, $S_3=4$.

If a sparse subset of $\{1,2,\ldots,n\}$ contains n, then it can not contain $n-1$ and $n-2$. On the other hand, union of every sparse subset of $\{1,2,\ldots,n-3\}$ with n is a sparse subset of $\{1,2,\ldots,n\}$. We also have that every sparse subset of $\{1,2,\ldots,n-1\}$ is also a sparse subset of $\{1,2,\ldots,n\}$. It follows that the number of sparse subsets of $\{1,2,\ldots,n\}$ that contain n is equal to S_{n-3}, while the number of sparse subsets of $\{1,2,\ldots,n\}$ that do not contain n is equal to S_{n-1}. Consequently, $S_n=S_{n-1}+S_{n-3}$. So the sequence S_n is

$$S_1=2, S_2=3, S_3=4, S_4=6, S_5=9, S_6=13,$$
$$S_7=19, S_8=28, S_9=41, S_{10}=60, S_{11}=88, S_{12}=129.$$

Answer: 129