EF Exam Solutions
Texas A&M High School Math Contest
November 4, 2023

All answers must be simplified, and if units are involved, be sure to include them.

1. Find $p+q$ if p and q are rational numbers such that $\sqrt{p+q\sqrt{7}} = \dfrac{9}{4-\sqrt{7}}$.

2. How many pairs of integers (x, y) are solutions of the equation $x^2 - xy + y^2 = x + y$.

3. Let f be an increasing function and g be a function such that
$$f(g(x) + 2023) \leq f(x) \leq (f \circ g)(x + 2023),$$
for all real numbers x. Find $g(0)$.

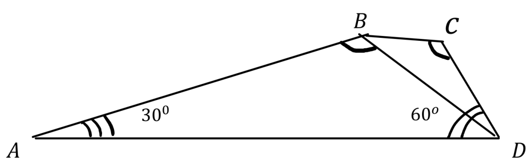
4. Find the exact value of
$$\sqrt{36^{\log_6 5} + 10^{1-\log 2} - 3^{\log_9 36} + 1}.$$

5. When the polynomial $P(x) = 3x^4 + mx^3 + nx^2 + 2x - 15$ is divided by $3x^2 + x - 2$, the remainder is $x - 5$. Find $m^2 + n^2$.

6. Consider the ellipse with vertices at $(0, -6)$ and $(0, 6)$ and passing through the point $(2, -4)$. Find the x-coordinate of the point where the ellipse intersects the positive x-axis.

7. Find the sum of all distinct real solutions of the equation
$$(3x^2 - 4x + 1)^3 + (x^2 + 4x - 5)^3 = 64(x^2 - 1)^3.$$

8. Consider the quadrilateral $ABCD$ with $BD = 10$, $BC = 5$, $m\angle BAD = 30°$, $m\angle CDA = 60°$, and $m\angle ABD = m\angle BCD$. Find CD.



9. Let f be a differentiable function such that $f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3 + 2h$ for all x and h and $f(0) = 1$. If $g(x) = e^{-x}f(x)$, find $g'(3)$.

10. Suppose that the lengths of the sides of a triangle are three consecutive integers. Find the perimeter of the triangle if we know that the perimeter is (numerically) half of the area of the triangle.

11. Find the length of the graph of the function $f(x) = \left(\dfrac{x}{2}\right)^2 - \ln\sqrt{x}$ on the interval $[1, e]$.

12. Find the sum (in radians) of the solutions in $[0, 2\pi]$ of the equation
$$(\sqrt{3}+1)\cos x + (\sqrt{3}-1)\sin x = 2.$$

13. Find the limit
$$L = \lim_{x\to\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}.$$

14. Point A is chosen at random from the line segment joining $(0,0)$ and $(2,0)$ as the center of a circle of radius 1. Point B is chosen at random from the line segment joining $(0,1)$ and $(2,1)$ as the center of another circle of radius 1. What is the probability that the two circles intersect?

15. Let x_1, x_2, and x_3 be the roots of the equation $x^3 - x + 1 = 0$. Find $x_1^{11} + x_2^{11} + x_3^{11}$.

16. Find $\theta \in (0, 180°)$ such that

$$\cot\theta = \frac{(1+\tan 1°)(1+\tan 2°)-2}{(1-\tan 1°)(1-\tan 2°)-2}.$$

17. Find the sum

$$\sum_{n=1}^{84} \frac{1}{\sqrt{2n+\sqrt{4n^2-1}}}.$$

18. Evaluate the limit

$$L = \lim_{n\to\infty}\left(\lim_{x\to 0}(1+\tan^2 x+\tan^2 2x+\cdots+\tan^2 nx)^{\frac{1}{n^3x^2}}\right).$$

19. Consider the sequence with general term

$$a_n = \sum_{k=1}^{n}\binom{n}{k}^2.$$

Find $\lim_{n\to\infty}\sqrt[n]{1+a_n}$.

20. Evaluate the integral

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2x \ln(1+2^{\sin x}+3^{\sin x}+6^{\sin x})dx.$$