

# TAMU 2015 Freshman-Sophomore Math Contest

## Solutions Sophomore Version

1. Let  $(X, Y)$  be the point nearest  $(1, 1)$  that's below the parabola  $y = x^2$  and at least 1 unit distant from every point on the parabola. Find  $X + 2Y$ .

The answer is 3. To find  $(X, Y)$ , we look for a point at distance 1 from  $(1, 1)$  that is below the parabola (so  $Y < X^2$ ) and that is not at a distance of 1 from any other point on the parabola. If we arrange that the circle of radius 1 about  $(X, Y)$  is tangent to the parabola at  $(1, 1)$ , that will happen. The slope of the tangent line to the parabola there is 2, so we want the slope of the radius from  $(1, 1)$  to  $(X, Y)$  to be  $-1/2$ . That means  $X = 1 + t$ ,  $Y = 1 - t/2$ , with  $t$  chosen so that  $t\sqrt{1 + 1/4} = 1$ . So  $t = 2/\sqrt{5}$ ,  $X = 1 + 2/\sqrt{5}$ ,  $Y = 1 - 1/\sqrt{5}$ , and  $X + 2Y = 3$ .

2. Let

$$f(x) = \frac{\log x}{1 + x^2}, \quad L = \int_1^{\infty} f(x) dx.$$

- (a) Prove that  $L$  is finite.

The answer to a question calling for a proof is a proof, so let's get started. Since the integrand is positive throughout its interval of integration, we only need to show that the improper integral does not diverge to infinity. One proof involves the integral comparison test. For  $x > 2$ ,  $\log x < \sqrt{x}$  because  $\log 2 < 2$  and because the derivative of  $\log x$  is  $1/x$  while the derivative of  $\sqrt{x}$  is  $(1/2\sqrt{x})$  which is larger.

Now, by the integral comparison test,  $\int_1^{\infty} f(x) dx < \int_1^4 f(x) dx + \int_4^{\infty} \sqrt{x}/x^2$ . The first part of this upper bound is finite because the interval of integration is finite, and the second part is finite because it evaluates to 1. QED.

Another proof involves integration by parts as well as comparison. First observe as in the previous proof that  $1/x^2 > 1/(x^2 + 1)$ . So we just need to prove that  $\int_1^{\infty} x^{-2} \log x dx$  is finite. Now take

$U = \log x$ ,  $dV = x^{-2}$ , which yields  $U = 1/x$  and  $V = -x^{-1}$ . Thus

$$\int_1^N \log x/x^2 dx = -\frac{\log x}{x} \Big|_1^N + \int_1^N x^{-2} dx.$$

As  $N$  tends to infinity, the evaluation part tends to zero and the integral part tends to 1. So the overall upper bound integral is 1. The actual integral is, of course, less.

- (b) Find a rational number  $A$  so that  $A < L < A + 1/25$ . In other words, find an approximation to  $L$  that pins  $L$  down to within  $\pm 0.04$ . Hint: For  $x > 1$ ,  $1/(x^2 + 1)$  can be written as a series of the form  $p_1/x^2 + p_2/x^4 + p_3/x^6 + \dots$ .

My favorite answer here is  $A = 8/9$ . Getting this involves first observing that the series is actually  $1/x^2 - 1/x^4 + 1/x^6 - 1/x^8 \dots$ . Thus

$$L = \sum_{n=0}^{\infty} (-1)^n \int_1^{\infty} \frac{\log x}{x^{2n+2}} dx.$$

These integrals evaluate in the same way as above, and the general answer is  $\pm 1/(2n + 1)^2$ , so

$$L = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \dots$$

This is an alternating series, so stopping right after a minus term gives us a lower bound, and stopping right after a plus sign gives us an upper bound. So  $1 - 1/9 < L < 1 - 1/9 + 1/25$  and  $A = 8/9$  will serve as an answer.

The actual number is known as Catalan's constant and often denoted  $G$ . Read all about it in the wikipedia article on *Catalan's constant*.

3. A thousand-meter deep borehole full of water has a tapered shape, with circular horizontal cross sections (whose centers are lined up vertically). The cross section of radius  $r$  is at depth  $1000 - r^3$ . (And  $r$  ranges from 0 to 10.)

- (a) Find the volume of the water in the borehole.

The volume is  $60000\pi$  cubic meters. This is the boring part. The volume is the integral of the depth. The depth at radius  $r$  is given

to be  $1000 - r^3$ . In polar coordinates, d area is given by  $r dr d\theta$ , or if everything is independent of  $\theta$ , as here, one just writes  $2\pi r dr$  and integrates only with respect to  $r$ . The volume (in cubic meters) is given by  $\int_{r=0}^{10} 2\pi r(1000 - r^3) dr = 60000\pi$ .

- (b) Find the work (in joules) required to lift all that water to the surface. ( $G = 9.8$ , water density = 1000 kg per cubic meter.)

All the units are MKS units, so we can dispense with keeping track of the names for the units. The work is the integral of mass times distance stuff has to be lifted, times  $G$ . A vertical straw reaching from the surface down to a depth of  $1000 - r^3$  meters will have volume  $1000 - r^3$  times its surface area, and while some of the water is not as deep as other parts of it, the average depth of the water in the straw is  $(1/2)(1000 - r^3)$ . Thus we set up the problem as

$$W = 9.8 \times 1000 \int_{r=0}^{10} 2\pi r \frac{1}{2} (1000 - r^3)^2 dr.$$

That comes to  $220500000000\pi$ . Just for your amusement and mine, I converted this into megawatt hours. It came to a bit under 200 of those. It would be expensive to empty that hole, even if the pump was 100 percent efficient.

4. Find the height of a regular tetrahedron with volume 1. (A regular tetrahedron is a solid with four vertices (corners), each connected to each of the others by edges all of the same length.) The required volume is  $2/3^{1/6}$ .

Consider a regular tetrahedron  $T$  with edges of length 1. The base is an equilateral triangle of edge 1. Taking one edge to be the interval  $[0, 1]$ , and the third vertex to have positive  $y$  coordinate, that third vertex is at  $(1/2, \sqrt{3}/2)$ . The centroid  $C$  of the base is thus  $(1/2, \sqrt{3}/6)$  and the distance to  $C$  from a vertex is  $\sqrt{1/4 + 3/36} = \sqrt{1/3}$ . The distance from  $C$  to the fourth vertex is the height  $H$  of  $T$ , so  $H = \sqrt{2/3}$  because the triangle including  $C$ , the opposite vertex, and any other vertex is a right triangle. Now the volume  $V$  of  $T$  is one third base area times height, so  $V = (1/3) \cdot \sqrt{2/3} \cdot \sqrt{3}/4 = \sqrt{2}/12$ .

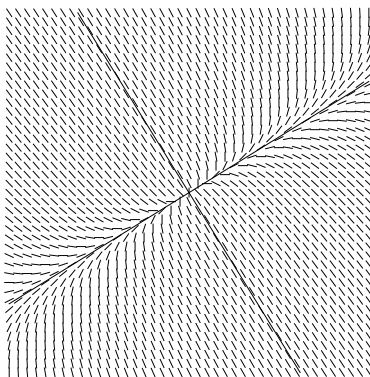
The tetrahedron whose height we want has volume 1, so we should scale  $T$  by a scaling factor of  $s = (12/\sqrt{2})^{1/3}$  in each of its three dimensions to bring its volume to 1. So the height of our unit-volume tetrahedron is  $s \cdot \sqrt{2/3}$  which works out to  $2 \cdot 3^{-1/6}$ .

5. Find

$$\int_{x=-1}^1 \int_{y=-1}^1 \frac{1}{1 + (x - y)^2} dy dx.$$

6. Consider the system of differential equations  $\frac{dx}{dt} = -5x + 2y$ ,  $\frac{dy}{dt} = -6x + 2y$ .

- (a) Sketch a direction field for the system, covering the region  $-2 \leq x, y \leq 2$ , and show some typical flows or paths the system can take.



- (b) Find the equations of two lines through the origin, each with the property that if the initial conditions  $x(0) = x_0$ ,  $y(0) = y_0$  give a point  $(x_0, y_0)$  on the line,  $(x(t), y(t))$  is on the line for all  $t$ . In other words, the solution moves along the line. The lines are  $y = m_1x$  and  $y = m_2x$ , where  $m_1$  and  $m_2$  are  $(-1 \pm \sqrt{5})/2$ .
- (c) If  $x_0 = y_0 = 1$ , find  $\lim_{t \rightarrow \infty} y(t)/x(t)$ . The limit is  $(\sqrt{5} - 1)/2$  because the flows converge toward the line of that slope and then, more or less along that line, toward the origin.