The Chowla-Selberg Formula for Quartic Abelian CM Fields

Robert Cass

University of Kentucky

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Classical Chowla-Selberg

Let $d \neq 1$ be a squarefree integer and let $K = \mathbb{Q}(\sqrt{d})$.

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- Discriminant Δ
- Ring of integers $\mathcal{O}_K$
- Ideal class group $\mathrm{CL}(K)$
- Class number $h_d = \#\,\mathrm{CL}(K)$
- Fundamental unit ϵ_d if $d > 0$
- Number of units $w_d = \#\mathcal{O}_K^\times$ if $d < 0$
- Kronecker symbol $\chi_d(k) = \left(\frac{\Delta}{k}\right)$

Classical Chowla-Selberg

- The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via linear fractional transformations.

$$\text{If } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad \text{then } \gamma z = \frac{az+b}{cz+d}.$$

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- The Dedekind eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}, \quad z \in \mathbb{H}$$

is a weight $1/2$ modular form.

Classical Chowla-Selberg

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- Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \quad \mathrm{Re}(s) > 0.$$

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Given $C \in \mathrm{CL}(K)$, there exists $\mathfrak{a} \in C$ and $\alpha, \beta \in K^\times$ such that

$$\mathfrak{a} = \mathbb{Z}\alpha + \mathbb{Z}\beta$$

and

$$z_{\mathfrak{a}} = \frac{\beta}{\alpha} \in \mathbb{H}.$$

The point $z_{\mathfrak{a}}$ is a CM point corresponding to C.

Theorem (Chowla-Selberg)

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(K)} G(z_{\mathfrak{a}}) = \left(\frac{1}{4\pi\sqrt{|\Delta|}}\right)^{\frac{h_d}{2}} \prod_{k=1}^{|\Delta|} \Gamma\left(\frac{k}{|\Delta|}\right)^{\frac{w_d \chi_d(k)}{4}} .$$

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Example. Let $K = \mathbb{Q}(\sqrt{-5})$. Then

$$G\left(\frac{\sqrt{-5}-1}{2}\right) \cdot G\left(\sqrt{-5}\right) = \frac{1}{8\pi\sqrt{5}} \left(\frac{\Gamma\left(\frac{1}{20}\right)\Gamma\left(\frac{9}{20}\right)\Gamma\left(\frac{11}{20}\right)\Gamma\left(\frac{19}{20}\right)}{\Gamma\left(\frac{3}{20}\right)\Gamma\left(\frac{7}{20}\right)\Gamma\left(\frac{13}{20}\right)\Gamma\left(\frac{17}{20}\right)}\right)^{\frac{1}{2}}.$$

Preview of Results

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- Our goal is to provide more explicit forms of their formula for quartic abelian CM fields.
- Consists of two main parts:
  1. Calculate the CM points at which we will evaluate a Hilbert modular function which generalizes $G(z)$.
  2. Identify families of quartic field extensions for which we can determine the analogue of the product of gamma values.

Definitions

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- Two types: biquadratic and cyclic.

Definitions

Definition

Let E be a CM field of degree $2n$. A **CM type** Φ for E is choice of n embeddings $\sigma_1, \ldots, \sigma_n : E \hookrightarrow \mathbb{C}$ with exactly one embedding chosen from each of the n pairs of complex conjugate embeddings of E.

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We will be interested in the points

$$E^\times \cap \mathbb{H}^n = \{z \in E^\times \mid \Phi(z) = (\sigma_1(z), \ldots, \sigma_n(z)) \in \mathbb{H}^n\}.$$

Definition

Let E be a CM field of degree $2n$ with totally real quadratic subfield F. Assume F has narrow class number one, and let Φ be CM type for E. For each $C \in \mathrm{CL}(E)$, there exists $\mathfrak{a} \in C$ and $\alpha, \beta \in E^\times$ such that

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta$$

and

$$z_\mathfrak{a} = \frac{\beta}{\alpha} \in E^\times \cap \mathbb{H}^n.$$

The point $z_\mathfrak{a}$ is a **CM point** corresponding to the ideal class C.

Algorithm

Implemented an algorithm in Sage to compute CM points for E quartic abelian.

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Example. Let $E = \mathbb{Q}(\sqrt{5}, \sqrt{-23})$. A complete set of CM points for the CM type $\{\mathrm{id}, \sigma\}$ where

$$\sigma(\sqrt{5}) = -\sqrt{5}, \quad \sigma(\sqrt{-23}) = \sqrt{-23}$$

is

$$z_1 = \frac{\sqrt{-23} - \sqrt{5}}{2}, \quad z_2 = \frac{\sqrt{-23} - 1}{4}, \quad z_3 = \frac{\sqrt{-23} - 2\sqrt{5} - 1}{4}.$$

The Hilbert Modular Function

- The group $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathbb{H}^n$. If F has real embeddings $\tau_1, \ldots, \tau_n$, then

$$\gamma z = (\tau_1(\gamma) z_1, \ldots, \tau_n(\gamma) z_n).$$

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- Let

$$H(z) = \sqrt{N(y)}\phi(z)$$

where $N(y) = \prod_{j=1}^{n} y_i$ is the product of the imaginary parts of the components of $z \in \mathbb{H}^n$ and $\phi(z)$ is yet to be defined.

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- The function $H \colon \mathbb{H}^n \to \mathbb{R}^+$ is $\mathrm{SL}_2(\mathcal{O}_F)$-invariant and generalizes $G(z)$.

The function $\phi\colon \mathbb{H}^n \to \mathbb{R}^+$ is defined by

$$\phi(z) = \exp\left(\frac{\zeta_F^*(-1)N(y)}{R_F} + \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} \frac{\sigma_1(\mu\partial_F)}{R_F \left| N_{F/\mathbb{Q}}(\mu\partial_F) \cdot N_{F/\mathbb{Q}}(\mu) \right|^{1/2}} e^{2\pi i T(\mu,z)} \right)$$

where R_F is the residue of the completed Dedekind zeta function $\zeta_F^*(s)$ at $s = 0$, ∂_F is the different of F,

$$\sigma_1(\mu\partial_F) = \sum_{\mathfrak{b} | \mu\partial_F} N_{F/\mathbb{Q}}(\mathfrak{b}),$$

and

$$T(\mu, z) = \sum_{j=1}^{n} \tau_j(\mu)x_j + i \sum_{j=1}^{n} |\tau_j(\mu)|y_j.$$

Analog of $\Gamma(s)$

As a consequence of the Bohr-Mollerup theorem, $f(x) = \log\left(\Gamma(x)/\sqrt{2\pi}\right)$ is the unique function $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that

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Let $g\colon \mathbb{R}^+ \to \mathbb{R}$ be the unique function such that

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Let $\Gamma_2(x) = \exp g(x)$, which is analogous to $\Gamma(x)/\sqrt{2\pi}$.

Main Results

Theorem (Biquadratic)

Let $p \equiv 1 \pmod 4$ *and* $q \equiv 3 \pmod 4$ *be primes. Let* $F = \mathbb{Q}(\sqrt{p})$ *and* $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, *and assume that* F *has narrow class number* 1. *Then*

$$\prod_{[\mathfrak{a}] \in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{8\pi q}\right)^{\frac{h_E}{2}} \prod_{k=1}^{q} \Gamma\left(\frac{k}{q}\right)^{\frac{h_E \chi_{-q}(k) w_{-q}}{4h_{-q}}} \times \prod_{k=1}^{pq} \Gamma\left(\frac{k}{pq}\right)^{\frac{h_E \chi_{-pq}(k) w_{-pq}}{4h_{-pq}}} \prod_{k=1}^{p} \Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E \chi_p(k)}{4\log(\epsilon_p)}}.$$

Main Results

Theorem (Cyclic)

Let $p \equiv 1 \pmod 4$ be a prime, and let $B, C > 0$ be integers such that $p = B^2 + C^2$ and $B \equiv 2 \pmod 4$. Let $F = \mathbb{Q}(\sqrt{p})$ and $E = \mathbb{Q}\left(\sqrt{-\left(p + B\sqrt{p}\right)}\right)$. If F has narrow class number 1, then

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{p}{8\pi\sqrt{|\Delta_E|}}\right)^{\frac{h_E}{2}} \times \prod_{k=1}^{p} \Gamma\left(\frac{k}{p}\right)^{-h_E \operatorname{Re} \frac{\chi(k)}{B_1(\chi)}} \prod_{k=1}^{p} \Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E \chi_p(k)}{4\log(\epsilon_p)}}.$$

Here χ is any choice of character of $(\mathbb{Z}/p\mathbb{Z})^\times$ that sends a primitive root modulo p to a primitive fourth root of unity and $B_1(\chi)$ is the first generalized Bernoulli number attached to χ.

Examples

Let $F = \mathbb{Q}(\sqrt{29})$. The field $E = \mathbb{Q}(\sqrt{29}, \sqrt{-31})$ is a biquadratic CM field. There are 21 CM points, and

$$\prod_{[\mathfrak{a}] \in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{248\pi}\right)^{\frac{21}{2}} \prod_{k=1}^{31} \Gamma\left(\frac{k}{31}\right)^{\frac{7}{2}\chi_{31}(k)}$$

$$\times \prod_{k=1}^{899} \Gamma\left(\frac{k}{899}\right)^{\frac{3}{4}\chi_{899}(k)} \prod_{k=1}^{29} \Gamma\left(\frac{k}{29}\right)^{\frac{21\chi_{29}(k)}{4\log\left(\frac{\sqrt{29}-5}{2}\right)}}.$$

Examples

$\frac{\sqrt{-31}-\sqrt{29}}{2}$	$\frac{\sqrt{-31}-1}{4}$	$\frac{\sqrt{-31}-2\sqrt{29}-1}{4}$
$\frac{\sqrt{-31}+\sqrt{29}}{-\sqrt{29}+7}$	$\frac{\sqrt{-31}-\sqrt{29}}{-\sqrt{29}+7}$	$\frac{\sqrt{-31}+\sqrt{29}}{-4\sqrt{29}+22}$
$\frac{\sqrt{-31}-\sqrt{29}}{-4\sqrt{29}+22}$	$\frac{\sqrt{-31}+5\sqrt{29}}{-3\sqrt{29}+17}$	$\frac{\sqrt{-31}-5\sqrt{29}}{-3\sqrt{29}+17}$
$\frac{\sqrt{-31}+5\sqrt{29}}{-2\sqrt{29}+12}$	$\frac{\sqrt{-31}-5\sqrt{29}}{-2\sqrt{29}+12}$	$\frac{\sqrt{-31}+\sqrt{29}}{6}$
$\frac{\sqrt{-31}-\sqrt{29}}{6}$	$\frac{\sqrt{-31}+4\sqrt{29}-1}{-2\sqrt{29}+14}$	$\frac{\sqrt{-31}+2\sqrt{29}-1}{-2\sqrt{29}+14}$
$\frac{\sqrt{-31}+19\sqrt{29}}{-7\sqrt{29}+39}$	$\frac{\sqrt{-31}-19\sqrt{29}}{-7\sqrt{29}+39}$	$\frac{\sqrt{-31}+19\sqrt{29}}{-3\sqrt{29}+19}$
$\frac{\sqrt{-31}-19\sqrt{29}}{-3\sqrt{29}+19}$	$\frac{\sqrt{-31}+10\sqrt{29}-1}{-6\sqrt{29}+34}$	$\frac{\sqrt{-31}-8\sqrt{29}-1}{-6\sqrt{29}+34}$

Examples

Let $F = \mathbb{Q}(\sqrt{13})$. Then the field $E = F\left(\sqrt{-(13+2\sqrt{13})}\right)$ is a cyclic CM field of degree 4 over $\mathbb{Q}$.

$$H\left(\frac{\sqrt{-(13+2\sqrt{13})}-3\sqrt{13}}{\sqrt{13}+5}, \frac{\sqrt{-(13-2\sqrt{13})}+3\sqrt{13}}{-\sqrt{13}+5}\right) =$$

$$\frac{1}{2\cdot 13^{\frac{1}{4}}\sqrt{2\pi}}\left(\frac{\Gamma\left(\frac{1}{13}\right)\Gamma\left(\frac{2}{13}\right)\Gamma\left(\frac{3}{13}\right)\Gamma\left(\frac{5}{13}\right)\Gamma\left(\frac{6}{13}\right)\Gamma\left(\frac{9}{13}\right)}{\Gamma\left(\frac{4}{13}\right)\Gamma\left(\frac{7}{13}\right)\Gamma\left(\frac{8}{13}\right)\Gamma\left(\frac{10}{13}\right)\Gamma\left(\frac{11}{13}\right)\Gamma\left(\frac{12}{13}\right)}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{\Gamma_2\left(\frac{1}{13}\right)\Gamma_2\left(\frac{3}{13}\right)\Gamma_2\left(\frac{4}{13}\right)\Gamma_2\left(\frac{9}{13}\right)\Gamma_2\left(\frac{10}{13}\right)\Gamma_2\left(\frac{12}{13}\right)}{\Gamma_2\left(\frac{2}{13}\right)\Gamma_2\left(\frac{5}{13}\right)\Gamma_2\left(\frac{6}{13}\right)\Gamma_2\left(\frac{7}{13}\right)\Gamma_2\left(\frac{8}{13}\right)\Gamma_2\left(\frac{11}{13}\right)}\right)^{\frac{1}{4\log\left(\frac{\sqrt{13}+3}{2}\right)}}.$$

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