MOMENTS OF L-FUNCTIONS ASSOCIATED TO NEWFORMS OF SQUAREFREE LEVEL

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Abstract. STILL TO DO!

Contents

1. Introduction

Moments of L-functions provide a powerful tool for studying arithmetic information. They have been used to study the nonvanishing of L-functions at the central point and subconvexity as well as other analytic properties of L-functions.

1.1. Preliminaries on Modular Forms and Trace Formulas.

Let $\mathcal{H}_k^*(q)$ denote the space of all newforms of squarefree level q and even integer weight κ. It is well known that this is a subspace of the (finite) vector space of modular forms of level q and weight κ. Further, when equipped with the Petersson inner product $\langle f, g\rangle := \int_{\gamma_0(q)\backslash\mathbb{H}} y^\kappa f(z)\overline{g(z)}\frac{dxdy}{y^2}$ A key tool in studying modular forms are trace formulas. Notably, the Petersson formula gives an orthgonality relation for fourier coefficients associated to modular forms that form a basis, for $\mathcal{S}_\kappa(q)$, the space of all cusp forms of weight κ and level q.

Let $\mathcal{B}$ be an orthogonal basis for $\mathcal{S}_\kappa(q)$, then define $\Delta_q(m,n) = c_\kappa \sum_{f\in\mathcal{B}} \frac{\lambda_f(m)\lambda_f(n)}{\langle f,f\rangle}$, where $c_\kappa = \frac{\Gamma(\kappa-1)}{(4\pi)^{\kappa-1}}$. By the Petersson formula, (regardless of our choice of $\mathcal{B}$ we have,

$$\Delta_q(m,n) = \delta(m=n) + 2\pi i^{-\kappa} \sum_{\substack{c>0\\ c\equiv 0(q)}} \frac{S(m,n;c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{*}$$

where $S(x,y;c)$ denotes the Kloosterman sum, and $J_{\kappa-1}(x)$ denotes the J-Bessel function of order $k-1$. For a proof of the Petersson Formula, see [5].

Let $\Delta_q^*(m,n) := \sum_{f\in\mathcal{H}_k^*(q)} \frac{\lambda_f(m)\lambda_f(n)}{\langle f,f\rangle}$. The starting point for this paper's calculations is the following orthogonality relation for newforms of squarefree level q, which do not generally form a basis for $\mathcal{S}_\kappa(q)$, due to Petrow and Young [9]:

$$\Delta_q^*(m,n) = \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\ell|L^\infty} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2|\ell} c_\ell(d_1)c_\ell(d_2) \sum_{\substack{u|(m,L)\\ v|(n,L)}} \frac{uv}{(u,v)} \frac{\mu(\frac{uv}{(u,v)^2})}{\nu(\frac{uv}{(u,v)^2})} \sum_{\substack{a|(\frac{m}{u},\frac{uv}{(u,v)})\\ b|(\frac{m}{u},\frac{uv}{(u,v)})}} \sum_{\substack{e_1|(d_1,\frac{m}{a^2(u,v)})\\ e_2|(d_2,\frac{n}{b^2(u,v)})}} \Delta_M(m,n), \tag{**}$$

where $c_\ell(d)$ is jointly multiplicative and $c_{p^n}(p^j) = c_{j,n}$ with $c_{j,n}$ such that

$$x^n = \sum_{j=0}^{n} c_{j,n} U_j\left(\frac{x}{2}\right),$$

where $U_n(x)$ denotes the n^{th} Chebyshev polynomial of the second kind.

Define,

$$A_q(n,m) := \sum_{L|q} \frac{\mu(L)}{\nu(L)} \sum_{\ell|L^\infty} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2|\ell} c_\ell(d_1)c_\ell(d_2) \sum_{\substack{u|(m,L)\\ v|(n,L)}} \frac{uv}{(u,v)} \frac{\mu(\frac{uv}{(u,v)^2})}{\nu(\frac{uv}{(u,v)^2})} \sum_{\substack{a|(\frac{m}{u},\frac{uv}{(u,v)})\\ b|(\frac{m}{u},\frac{uv}{(u,v)})}} \sum_{\substack{e_1|(d_1,\frac{m}{a^2(u,v)})\\ e_2|(d_2,\frac{n}{b^2(u,v)})}} \delta(m=n).$$

By applying the Weil bound applied to the Kloosterman sums and the bound $J_{\kappa-1}(z) \ll_\kappa \min(1,z)$, we see that $\Delta_q^*(m,n) - A_q(n,m) = O_\kappa(q^{-1+\epsilon}(mn)^{\frac{1}{4}+\epsilon})$.

The first main result of this paper is the following explicit formula for $A_q(n,m)$.

Theorem 1.1 (Approximate Orthogonality of Newforms). *Let $m = \prod_p p^{m_i}$ and $n = \prod_p p^{n_i}$, then*

$$A_q(n,m) = \begin{cases} \frac{\phi(q)}{q} \prod_{p|q} \sum_{\substack{n \le n_i \\ m \le m_i}} p^{-\frac{m_i+n_i}{2}} \prod_{p \nmid q} \delta(m_i = n_i) & \text{if } mn \text{ is square,} \\ 0 & \text{otherwise.} \end{cases}$$

1.2. **Preliminaries on L-functions associated to newforms.** To each newform, we can associate an L-function in a natural way. In particular, let $f \in \mathcal{H}_k^*(q)$ and consider the Fourier expansion of f, so $f(z) = \sum_{n=0}^{\infty} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz)$. Then to f we associate the L-function,

$$L(s,f) = \sum_{n=0}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} - \frac{\chi_0(p)}{p^{2s}}\right)^{-1}.$$

where χ_0 is the trivial character $\pmod q$.

This series converges absolutely for $s > 1$, and can be extended to an entire function. Further, we define the completed L-function,

$$\Lambda(s,f) = \left(\frac{\sqrt{q}}{2\pi}\right)^{s+\frac{\kappa-1}{2}} \Gamma(s + \frac{\kappa-1}{2}) L(s,f), \tag{1}$$

which satisfies the symmetric functional equation, $\Lambda(s,f) = \epsilon_f \Lambda(1-s,f)$ where $\epsilon_f = \pm 1$ is the root number of f. The average value of $L(\frac{1}{2}, f)$ at the central point is the focus of this paper. Precisely, define

$$\mathcal{M}^{(t)}_{\alpha_1,\alpha_2,\ldots,\alpha_t} := \sum_{f \in \mathcal{H}_k^*(q)} \omega_f \prod_{i=1}^{t} L(\frac{1}{2} + \alpha_i, f),$$

where $\omega_f := \frac{c_\kappa}{\langle f,f \rangle}$ Then the following two theorems giving the asymptotic for $\mathcal{M}^t$ for $t = 1,2$ are the main results of this paper:

Theorem 1.2 (First Moment). *Let α satisfy $|Re(\alpha)| < \frac{1}{2}$ and for all $\epsilon > 0$, $Im(\alpha) \ll q^\epsilon$. Define $\gamma = \min(0, Re(\alpha))$ then*

$$\mathcal{M}^{(1)}_\alpha = \frac{\phi(q)}{q} \prod_{p|q} \left(\frac{1}{(1-p^{-(2+2\alpha)})}\right) + O(q^{-1-\gamma+\epsilon}).$$

where the implied constant depends on k and ϵ.

Theorem 1.3 (Second Moment). *With ω_f as before and α, β shifts with real part less than $1/2$ in absolute value, and imaginary part bounded be q^ϵ for all $\epsilon > 0$ then for any $\epsilon > 0$,* *we have*

$$\mathcal{M}^{(2)}_{\alpha,\beta} = \frac{\phi(q)}{q}\Bigg(\zeta(1+\alpha+\beta)\prod_{p|q}\frac{(1+p^{-2-\alpha-\beta})(1-p^{-1-\alpha-\beta})}{(1-p^{-2(1+\alpha)})(1-p^{-2(1+\beta)})}+$$
$$\left(\frac{2\pi}{\sqrt{q}}\right)^{2(\alpha+\beta)}\frac{\Gamma(\alpha+\frac{\kappa}{2})\Gamma(\beta+\frac{\kappa}{2})}{\Gamma(-\alpha+\frac{\kappa}{2})\Gamma(-\beta+\frac{\kappa}{2})}\zeta(1-\alpha-\beta)\prod_{p|q}\frac{(1+p^{-2+\alpha+\beta})(1-p^{-1+\alpha+\beta})}{(1-p^{-2(1-\alpha)})(1-p^{-2(1-\beta)})}\Bigg)+$$
$$O(q^{-\frac{1}{2}-\min(Re(\alpha),Re(\beta))+\epsilon}) \quad (2)$$

where the implied constant depends on κ *and* ϵ.

Duke [4] was the first to compute the first moment and obtain an upper bound on the second moment at the central point for L-functions associated to newforms, with the restriction of prime level and weight 2. Akbary [1] obtained the same result for general weights. Kowalski, Michel and Vanderkam [7] were the first to obtain an asymptotic for the fourth moment and used a mollified moment to prove a positive percentage of nonvanishing at the central point. Rouymi [11] generalized the work of Duke by obtaining asymptotics for the first three moments for arbitrary weight and prime power level. Balkanova [2] obtained an asymptotic for the fourth moment in the case of prime power level. In this paper, we generalize previous results in a different direction, by obtaining asymptotics for the first two moments with arbitrary weight and squarefree level. [3] conjectured asymptotics for all even t. However, Theorem 1.3 does not agree with their conjecture, as it appears they omitted the arithmetic factor coming from primes dividing the level. Following their recipe for conjectural moments utilizing 1.1 could give improved conjectural moments for even t.

2. Proof of Approximate Orthogonality

We first note that $A_q(n,m)$ is jointly multiplicative. Further, note that $A_1(p^m,p^n) = \delta(m=n)$, so it is suffices to prove the following lemma:

Proposition 2.1. *Let* p *be a prime, and* m,n *nonnegative integers then*

$$A_p(p^m,p^n) = \begin{cases} \frac{\phi(p)}{p}p^{-\frac{m+n}{2}} & m \equiv n \pmod 2, \\ 0 & \textit{otherwise}. \end{cases}$$

The first case can be proven directly using properties of the λ_f. In particular, it follows from the complete multiplicativity of Fourier coefficients at primes dividing the level, and the work of Winnie Li, [8] who showed that for primes dividing the level, $|\lambda_f(p)| = \frac{1}{\sqrt{p}}$, then taking limits appropriately so that $\Delta^*_q(m,n)$ converges to $A_q(n,m)$. However, we instead work directly from (**). We first prove Proposition 2.1 contingent on the following two lemmas:

Lemma 2.2. *For* $m \geq n \geq 0$, *define*

$$S_\ell(m,n) := \sum_{d_1,d_2\leq\ell}\sum_{\substack{e_1\leq\min(d_1,m)\\ e_2\leq\min(d_2,n)}} c_{d_1,\ell}c_{d_2,\ell}\delta\Big(m+d_1-2e_1 = n+d_2-2e_2\Big).$$

Then,

$$S_\ell(m,n) = \begin{cases} \binom{2\ell}{\ell-\frac{m-n}{2}} - \binom{2\ell}{\ell-\frac{m-n}{2}-n-1} & m \equiv n \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Lemma 2.3. *Let $a \geq 0$ an integer, and $|x| < \frac{1}{4}$. Then*

$$F_a(x) := \sum_{k=0}^{\infty} \binom{2k+a}{k} x^k = \frac{(\frac{1-\sqrt{1-4x}}{2x})^a}{\sqrt{1-4x}}. \tag{4}$$

In particular, if $x = \frac{p}{(p+1)^2}$ with $p \geq 2$, the RHS simplifies to $\frac{(p+1)^{a+1}}{(p-1)p^a}$.

Proof of 2.1. From its definition, we have

$$A_p(p^m,p^n) = \delta(m=n) - \frac{1}{p+1}\Big(B_{0,0,0,0}(m,n) + \delta(n>0)B_{1,1,0,0}(m,n) + \delta(m>0)B_{1,0,0,0}(m,n) + \delta(m>1)B_{1,0,1,0}(m,n) + \delta(n>0)B_{0,1,0,0}(m,n) + \delta(n>1)B_{0,1,0,1}(m,n)\Big), \tag{5}$$

where

$$B_{u',v',a',b'}(m,n) := \sum_{\ell=0}^{\infty}\left(\frac{p}{(p+1)^2}\right)^{\ell} \sum_{d_1,d_2\leq \ell} p^{u'+v'-\min(u',v')}\frac{\mu(p^{u+v-2\min(u',v')})}{\nu(p^{u'+v'-2\min(u',v')})} \sum_{\substack{e_1\leq \min(d_1,m-2a'-\min(u',v'))\\ e_2\leq\min(d_2,n-2b'-\min(u',v'))}} c_{d_1,\ell}c_{d_2,\ell}\delta\Big(m+d_1-2(a'+e_1) = n+d_2-2(b'+e_2)\Big).$$

Under the change of variables $m \to m-2a'-\min(u',v')$ and $n \to n-2b'-\min(u',v')$, we have

$$B_{u',v',a',b'} = p^{u'+v'-\min(u',v')}\frac{\mu(p^{u'+v'-2\min(u',v')})}{\nu(p^{u'+v'-2\min(u',v')})}B_{0,0,0,0}(m-2a'-\min(u',v'), n-2b'-\min(u',v')),$$

so that (5) becomes

$$A_p(p^m,p^n) = \delta(m,n) - \frac{1}{p+1}\Big(B(m,n) + \delta(n>0)pB(m-1,n-1) - \frac{p}{p+1}\delta(m>0)B(m,n) - \frac{p}{p+1}\delta(m>1)B(m-2,n) - \frac{p}{p+1}\delta(n>0)B(m,n) - \frac{p}{p+1}\delta(n>1)B(m,n-2)\Big)$$

with $B = B_{0,0,0,0}$. We have $B(m,n) = \sum_{\ell=0}^{\infty}\left(\frac{p}{(p+1)^2}\right)^{\ell} S_\ell(m,n)$, which by Lemma 2.2 and applying Lemma 2.3 to each of the resulting binomial coefficients under a suitable change of variables gives Lemma 2.1. □

Proof of 2.3. We begin with the identity $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{j}{k}x^k y^j = \frac{1}{1-y-xy}$, which is a direct consequence of the binomial theorem and geometric series formula after interchanging sums (the inner sum being finite). Then,

$$F_a(x) = \sum_k \binom{2k+a}{k} x^k = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{j}{k} x^k \delta(j, 2k+a) = \frac{1}{2\pi i}\sum_{k,j}\binom{j}{k} x^k \int_\gamma z^{-1+(2k+a-j)} dz,$$

where γ is a circle centered around the origin of radius $r = \frac{1}{2x}$. After interchanging the order of summation and integration, and applying the geometric series formula, we have,

$$F_a(x) = \frac{1}{2\pi i}\int_\gamma \frac{z^a}{z-1-xz^2} dz.$$

Note that for z inside the curve, and by our assumption that $|x| < 1/4$ we have $|1/z + xz| < 2|x| + \frac{1}{2} < 1$ so the geometric series converges. By our assumption that $a \geq 0$, and $x < 1/4$ we have a single pole inside the circle at $z = \frac{1-\sqrt{1-4x}}{2x}$. This gives a residue of,

$$\frac{(\frac{1-\sqrt{1-4x}}{2x})^a}{\sqrt{1-4x}}.$$

as claimed. □

To complete the proof of the orthogonality relation, it remains to prove Lemma 2.2, the proof of which will be the contents of the next section.

3. Chebyshev Polynomials and Coefficients

In proving the explicit formula for the $S_\ell(m,n)$, (2.2), we need several basic properties of Chebyshev polynomials which we gather in the next few lemmas.

Lemma 3.1 (Product Rule for Chebyshev Polynomials)**.** *For integers $m, n \geq 0$ we have,*

$$U_m(x)U_n(x) = \sum_{e=0}^{\min(m,n)} U_{m+n-2e}(x).$$

Lemma 3.2 (Orthogonality of Chebyshev Polynomials)**.** *For m, n integers we have,*

$$\int_{-2}^{2} U_m\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}} dx = \delta(m=n).$$

Lemma 3.3 (Integral Representation of Catalan Numbers)**.** *For integer $n > 0$ we have,*

$$C_n := \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}} dx.$$

The proof of 3.3 follows from rewriting the Catalan number as a ratio of gamma functions.

Lemma 3.4 (Integral Representation for $S_\ell(m,n)$)**.** *For $m \geq n \geq 0$ the $S_\ell(m,n)$ have the following integral representation:*

$$S_\ell(m,n) = \int_{x=-2}^{2} U_m\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) x^{2\ell} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx.$$

Proof. We have,

$$x^{2\ell} = \sum_{d_1,d_2 \leq \ell} c_{d_1,\ell} c_{d_2,\ell} U_{d_1}\left(\frac{x}{2}\right) U_{d_2}\left(\frac{x}{2}\right),$$

so

$$U_m\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) x^{2\ell} = \sum_{d_1,d_2 \leq \ell} c_{d_1} c_{d_2} U_{d_1}\left(\frac{x}{2}\right) U_m\left(\frac{x}{2}\right) U_{d_2}\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) x^{2\ell}.$$

Using Lemma 3.1,

$$U_m\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) x^{2\ell} = \sum_{d_1,d_2 \leq \ell} \sum_{\substack{e_1 \leq \min(m,d_1) \\ e_2 \leq \min(n,d_2)}} c_{d_1} c_{d_2} U_{m+d_1-2e_1}\left(\frac{x}{2}\right) U_{n+d_2-2e_2}\left(\frac{x}{2}\right).$$

Integrating both sides from $x=-2$ to 2 with respect to the measure $d\mu(x) = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx$, and using Lemma 3.4 to simplify the RHS,

$$\int_{-2}^{2} U_m\left(\frac{x}{2}\right) U_n\left(\frac{x}{2}\right) x^{2\ell} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx = S_\ell(m,n). \qquad \square$$

Lemma 3.5. *For $m \geq 0$ even and integer $\ell \geq \frac{m}{2}$ the following equalities hold:*

$$\int_{-2}^{2} U_m\left(\frac{x}{2}\right) x^{2\ell} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx = \binom{2\ell}{\ell-\frac{m}{2}} - \binom{2\ell}{\ell-\frac{m}{2}-1},$$

and

$$\int_{-2}^{2} U_{m+1}\left(\frac{x}{2}\right) x^{2\ell+1} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx = \binom{2\ell}{\ell-\frac{m}{2}} - \binom{2\ell}{\ell-\frac{m}{2}-2}$$

Proof: We proceed by induction on m using the recurrence $U_{m+2}\left(\frac{x}{2}\right) = xU_{m+1}\left(\frac{x}{2}\right) - U_m\left(\frac{x}{2}\right)$. We therefore need to establish the two base cases when $m=0$ and when $m=1$.

Base Case 1: $m=0$

Taking the change of variables $x=y^2$,

$$S_\ell(0,0) = \int_{x=-2}^{2} x^{2\ell} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx = \frac{1}{2\pi}\int_0^4 y^\ell \sqrt{\frac{4-y}{y}}dy = C_\ell = \binom{2\ell}{\ell} - \binom{2\ell}{\ell-1}.$$

Base Case 2: $m=1$

$$\int_{x=-2}^{2} x^{2\ell+1} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx = \frac{1}{2\pi}\int_0^4 y^{\ell+1}\sqrt{\frac{4-y}{y}}dy = C_{\ell+1} = \binom{2\ell+2}{\ell+1} - \binom{2\ell+2}{\ell} = \binom{2\ell}{\ell} - \binom{2\ell}{\ell-2}.$$

Even Inductive Step:

$$\begin{aligned}\int_{x=-2}^{2} U_m\left(\frac{x}{2}\right)U_n\left(\frac{x}{2}\right)x^{2\ell}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx &= \int_{x=-2}^{2} U_{m-1}\left(\frac{x}{2}\right)x^{2\ell+1}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx \\ &+ \int_{x=-2}^{2} U_{m-2}\left(\frac{x}{2}\right)x^{2\ell}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx \\ &= \binom{2\ell}{\ell-\frac{m}{2}+1}-\binom{2\ell}{\ell-\frac{m}{2}-1}-\left(\binom{2\ell}{\ell-\frac{m}{2}+1}-\binom{2\ell}{\ell-\frac{m}{2}}\right)\end{aligned}$$

giving the claimed equality.

Odd Inductive Step:

$$\begin{aligned}\int_{x=-2}^{2} U_{m+1}\left(\frac{x}{2}\right)x^{2\ell+1}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx &= \int_{x=-2}^{2} U_m\left(\frac{x}{2}\right)U_n\left(\frac{x}{2}\right)x^{2\ell+2}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx \\ &+ \int_{x=-2}^{2} U_{m-1}\left(\frac{x}{2}\right)x^{2\ell+1}\frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx \\ &= \binom{2\ell+2}{\ell-\frac{m}{2}+1}-\binom{2\ell+2}{\ell-\frac{m}{2}}-\binom{2\ell}{\ell-\frac{m}{2}+1}+\binom{2\ell}{\ell-\frac{m}{2}-1} \\ &= \binom{2\ell}{\ell-\frac{m}{2}}-\binom{2\ell}{\ell-\frac{m}{2}-2}\end{aligned}$$

From Lemma 3.1 applied to Lemma 3.3 we have $S_\ell(m,n)=\sum_{d=m-n,d\text{ even}}^{m+n} S_\ell(d,0)$ and the case for general n follows from noting the series

$$\sum_{d=m-nd\text{ even}}^{m+n}\binom{2\ell}{\ell-\frac{m}{2}}-\binom{2\ell}{\ell-\frac{m}{2}-1},$$

telescopes to $\binom{2\ell}{\ell-\frac{m-n}{2}}-\binom{2\ell}{\ell-\frac{m+n}{2}-1}$.

It is interesting to note the $c_{i,j}$ are linked to the Catalan numbers, a sequence defined recursively by $C_0=1$ and $C_n=\sum_{j=0}^{n} C_jC_{n-j}$. The Catalan numbers can also be viewed as the lead diagonal elements in Catalan's Triangle (See Fig. 1). Catalan's triangle is a triangular array defined recursively, where the left most column is identically 1 and each entry is the sum of the entry to its left and the one above it (if there is no entry above it, it is simply equal to the entry to its left).. Then we have $c_{i,j}=a_{(i+j)/2,(i-j)/2}$ where $a_{n,k}$ is the entry in the n^{th} row and k^{th} column of the Catalan triangle. The entries in Catalan's Triangle encode a variety of combinatorial information. See [12] for an extensive list. For example, $a_{n,k}$ is equal to the number of paths below the line $y=x$ on an integer lattice from the origin to (n,k) in which one can only move up and to the right.

$$\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 5 \\
1 & 4 & 9 & 14 & 14 \\
1 & 5 & 14 & 28 & 42 & 42 \\
1 & 6 & 20 & 48 & 90 & 132 & 132 \\
1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

FIGURE 1. Catalan's Triangle (equiv. Catalan's Trapezoid with n=1)

$$\begin{array}{ccccccccc}
1 & 1 & 1 \\
1 & 2 & 3 & 3 \\
1 & 3 & 6 & 9 & 9 \\
1 & 4 & 10 & 19 & 28 & 28 \\
1 & 5 & 15 & 34 & 62 & 90 & 90 \\
1 & 6 & 21 & 55 & 117 & 207 & 297 & 297 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

FIGURE 2. Catalan's Trapezoid with n=3

A natural generalization of Catalan's triangle is Catalan's trapezoid, in which the top row is a sequence of n $1's$, and where the same recurrence relation is satisfied. In this way, Catalan's triangle is simply the first Catalan trapezoid. Then we have that $S_\ell(m,n) = C_{n+1}(\ell + \frac{m-n}{2}, \ell - \frac{m-n}{2})$, where $C_i(j,k)$ denotes the entry in the j^{th} row and k^{th} column of the i^{th} Catalan Trapezoid. This can be proven directly via showing the boundary conditions are satisfied and the $S_\ell(m,n)$ satisfy the same recurrence as Catalan's trapezoid, though the proof is somewhat more involved and less intuitive than the proof given here for the explicit formula for the $S_\ell(m,n)$. It is interesting to compare Lemma 2.2 and [10, (3.4)], which links a different form of iterated sums to the Catalan Trapezoid.

4. First Moment

The main ingredients in the calculation of the first moment will be an asymmetric approximate functional equation for $L(s,f)$ and the approximate orthogonality relation.

Lemma 4.1 (Theorem 5.3 [6])**.**

$$L(\frac{1}{2}+\alpha, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2}+\alpha}} V_{\frac{1}{2}+\alpha}\left(\frac{n}{X}\right) + \epsilon_f \left(\frac{\sqrt{q}}{2\pi}\right)^{\alpha} \frac{\Gamma(u+\frac{k-1}{2}+s)}{\Gamma(u+\frac{k-1}{2})} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}} V_{\frac{1}{2}-\alpha}\left(\frac{nX}{q}\right) =: \mathcal{S}_1 + \mathcal{S}_2.$$

where

$$V_u(x) := \frac{1}{2\pi i}\int_{(3)} x^{-s} g_u(s) \frac{G(s)ds}{s},$$

$$g_u(s) := \frac{\Gamma(1-u+\frac{k-1}{2})}{\Gamma(u+\frac{k-1}{2})},$$

and $\epsilon_f = \pm 1$ is the root number of f, and $G(s)$ is an even, entire function of not more than polynomial growth.

We follow Duke [4] in taking $X = q^{1+\epsilon}$ so that the second term is small compared to the first to avoid the complexity introduced the root number.

Corollary 4.2 (Dirichlet series for $A_q(n,m)$)**.** *Let $m = \prod_p p^{m_i}$ and $n = \prod_p p^{n_i}$, then*

$$\sum_{n,m} \frac{A_q(n,m)}{n^s m^u} = \frac{\phi(q)}{q}\zeta(s+u)\prod_{p|q}\frac{(1+p^{-1-s-u})(1-p^{-s-u})}{(1-p^{-(2s+1)})(1-p^{-(2u+1)})}.$$

Proof of Corollary 4.2. We have,

$$\sum_{n,m}\frac{A_q(n,m)}{n^s m^u} = \frac{\phi(q)}{q}\prod_{p|q}\sum_{\substack{n,m\\ n+m \text{ even}}}\frac{p^{-\frac{m+n}{2}}}{p^{ns}p^{mu}}\prod_{p\nmid q}\sum_{n,m}\frac{\delta(m=n)}{p^{ns}p^{mu}}.$$

The second part of the equation is simply a geometric series. Splitting the first part of the equation into the cases based on the parity of m and n and noting that the resulting sums are the same under the change of variables, $m \to m+1, n \to n+1$, then applying the geometric series formula gives,

$$\sum_{n,m}\frac{A_q(n,m)}{n^s m^u} = \frac{\phi(q)}{q}\prod_{p|q}(1+p^{-1-s-u})\frac{1}{1-p^{-(2s+1)}}\frac{1}{1-p^{-(2u+1)}}\prod_{p\nmid q}\frac{1}{1-p^{-s-u}}$$

The claimed formula follows from factoring out a $\zeta(s+u)$ which agrees with the formula at primes not dividing the level. □

Corollary 4.3. *Let $n = \prod_p p^{n_i}$. Then,*

$$\sum_{n>0}\frac{A_q(n,1)}{n^s} = \frac{\phi(q)}{q}\prod_{p|q}\frac{1}{(1-p^{-(2s+1)})}$$

Proof. This follows from Corollary 4.2 taking the limit as $u \to \infty$ in the real direction. □

Recall, $\mathcal{M}_\alpha^{(1)} = \sum_{f\in\mathcal{H}_k^*(q)} \omega_f(\mathcal{S}_1 + \mathcal{S}_2)$. The main term will come from $\mathcal{S}_1$ which we consider first. We have,

$$\sum_{f\in\mathcal{H}_k^*(q)}\omega_f\mathcal{S}_1 = \sum_{n>0}\frac{A_q(n,1)+\mathcal{E}_q(n,1)}{n^{\frac{1}{2}+\alpha+s}}V\left(\frac{n}{X}\right) = \mathcal{A}_1 + \mathcal{E}_1.$$

Lemma 4.4. *We have,*

$$\mathcal{A}_1 = \frac{\phi(q)}{q} \prod_{p|q} \left(\frac{1}{(1-p^{-(2+2\alpha)})} \right) + O(q^{-1-Re(\alpha)+\epsilon}).$$

Opening up the definition of V gives us,

$$\mathcal{A}_1 = \frac{1}{2\pi i} \int_{(3)} \sum_{n=1}^{\infty} \frac{A_q(n,1)}{n^{\frac{1}{2}+\alpha}} \left(\frac{n}{X} \right)^{-s} \frac{G(s)ds}{s},$$

where we take $X = q^{1+\epsilon}$.

By Corollary 4.3 evaluated at $\frac{1}{2}+\alpha+s$, this becomes

$$\frac{1}{2\pi i} \frac{\phi(q)}{q} \int_{(3)} \prod_{p|q} \left(\frac{1}{(1-p^{-(2+2\alpha+2s)})} \right) \left(q^{1+\epsilon}\right)^s \frac{G(s)ds}{s}.$$

Shifting contours to $(-1-Re(\alpha)+\epsilon)$ we pick up a residue at $s=0$. We then may write,

$$\mathcal{A}_1 = R + J$$

where

$$R = \frac{\phi(q)}{q} \prod_{p|q} \left(\frac{1}{(1-p^{-(2+2\alpha)})} \right),$$

$$J = \frac{1}{2\pi i} \frac{\phi(q)}{q} \int_{(-1-Re(\alpha)+\epsilon)} \prod_{p|q} \left(\frac{1}{(1-p^{-(2+\alpha+s)})} \right) \left(q^{1+\epsilon}\right)^s \frac{G(s)ds}{s}.$$

Since for s on this line, we have

$$\left| \prod_{p|q} \left(\frac{1}{(1-p^{-(2+\alpha+s)})} \right) \right| < q^{\epsilon}.$$

We conclude,

$$J = O(q^{-1-Re(\alpha)+\epsilon}).$$

Finally, $\mathcal{S}_2 = O(q^{-100})$ using that for all A, $V(1/x) \ll_A (1+x)^{-A}$. This completes the proof of 4.4.

4.1. **Poisson Summation in One Variable.** The following two lemmas of [9] that allow us to bound sums of the $c_d(\ell)$ will be essential.

Lemma 4.5 (Theorem 7.1 of [9])**.**

$$\Delta_q^*(m,n) = \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\substack{\ell \mid L^\infty \\ \ell < Y}} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2 \mid \ell} c_\ell(d_1) c_\ell(d_2) \sum_{\substack{u \mid (m,L) \\ v \mid (n,L)}} \frac{uv}{(u,v)} \frac{\mu(\frac{uv}{(u,v)^2})}{\nu(\frac{uv}{(u,v)^2})}$$
$$\sum_{\substack{a \mid (\frac{m}{u}, \frac{uv}{(u,v)}) \\ b \mid (\frac{m}{u}, \frac{uv}{(u,v)})}} \sum_{\substack{e_1 \mid (d_1, \frac{m}{a^2(u,v)}) \\ e_2 \mid (d_2, \frac{n}{b^2(u,v)})}} \Delta_q\left(\frac{md_1}{a^2 e_1^2 (u,v)}, \frac{d_2}{b^2 e_2^2 (u,v)}\right) + O((mnqY)^\epsilon q Y^{-2\gamma_0})$$

where $\gamma_0 = \frac{\log(3/2)}{\log(2)} - \frac{1}{2}$. *In practice, we see that this error term can be made small if we take* Y *to be a sufficiently large power of* q.

and

Lemma 4.6. *Define* $S(L,Y) = \sum_{\substack{\ell \mid L^\infty \\ \ell < Y}} \frac{\ell}{\nu(\ell)^2} \left(\sum_{d \mid \ell} c_\ell(d) d^{1/2}\right)^2$ *Then,*

$$S(L,Y) \ll Y^\epsilon.$$

This is [9, 6.14].

By applying Lemma 4.5 and Lemma 4.6, pulling the resulting sum through up to the sum over n, and using that for $n \gg q^{1+\epsilon}$, we have rapid decay from V we get,

$$\mathcal{E}_1 = 2\pi i^{-k} \sum_{n=1}^\infty \frac{1}{n^{1/2+\alpha}} \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\substack{\ell \mid L^\infty \\ \ell \le Y}} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2 \mid \ell} c_\ell(d_1) c_\ell(d_2) \sum_{u \mid (n,L)} u \frac{\mu(u)}{\nu(u)} \sum_{b \mid (\frac{n}{u}, u)}$$
$$\sum_{e_2 \mid (d_1, \frac{n}{b^2 u})} \sum_{\substack{c \equiv 0 (M) \\ c > 0}} \frac{S(n',1;c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{n'}}{c}\right) V_{\frac{1}{2}+\alpha}\left(\frac{n}{q^{1+\epsilon}}\right) + O((qY)^\epsilon q Y^{-2\gamma_0}),$$

where $n' = \frac{nd_1}{b^2 e_2^2}$. The main goal of this section is the following proposition:

Proposition 4.7. *Let* $\gamma = \max(0, Re(\alpha))$ *then,*

$$\mathcal{E}_1 = O(q^{-1-\gamma+\epsilon}).$$

Via elementary arguments following Section 8 of [9], we have

$$\mathcal{E}_1 = 2\pi i^{-k} \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\substack{\ell \mid L^\infty \\ \ell \le Y}} \frac{\ell}{\nu(\ell)^2} \sum_{d_1,d_2 \mid \ell} c_\ell(d_1) c_\ell(d_2) \sum_{u \mid L} u \frac{\mu(u)}{\nu(u)} \sum_{a \mid u}$$
$$\sum_{e_1 \mid d_1} \sum_{\substack{0 < n \\ n \equiv 0 (D)}} \sum_{\substack{c \equiv 0 (M) \\ c > 0}} \frac{S(\delta n',1;c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{\delta n'}}{c}\right) V_{\frac{1}{2}+\alpha}\left(\frac{n}{q^{1+\epsilon}}\right) + O((qY)^\epsilon q Y^{-2\gamma_0})$$

where $D = au\frac{e_1}{(e_1, \frac{u}{a})}$.

Simplifying a bit further, let $A = \frac{d_1 u (e_1, \frac{u}{a})}{ae_1}$, Then,

$$\mathcal{E}_1 = 2\pi i^{-k} \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\substack{\ell | L^\infty \\ \ell \le Y}} \frac{\ell}{\nu(\ell)^2} \sum_{d_1, d_2 | \ell} c_\ell(d_1) c_\ell(d_2) \sum_{u|L} u \frac{\mu(u)}{\nu(u)} \sum_{a|u} \sum_{e_1 | d_1} S' + O((qY)^\epsilon q Y^{-2\gamma_0})$$

where

$$S' := \sum_{\substack{c \equiv 0(M) \\ c>0}} \sum_{n \ge 1} \frac{S(An, 1; c)}{n^{\frac{1}{2}+\alpha} c} J_{\kappa-1}\left(\frac{4\pi\sqrt{An}}{c}\right) V_{\frac{1}{2}+\alpha}\left(\frac{Dn}{q^{1+\epsilon}}\right). \qquad (***)$$

Proposition 4.8. *With S' as in (***),*

$$S' = O(M^{-1}\sqrt{A} q^{-\gamma+\epsilon})$$

For $c \gg Mq$ we have via the Weil bound

$$S' \ll \sum_{\substack{c \equiv 0(M) \\ c > qM}} \sum_{n < q^{1+\epsilon}} c^{-\frac{3}{2}+\epsilon} \sqrt{An} + O(q^{-100}) \ll \frac{q^\epsilon}{M}\sqrt{A},$$

which along with Lemma 4.6 gives the claimed bound. It remans to consider $c \ll Mq$.

We apply a dyadic partition of unity so that $w_N(x)$ is compactly supported on $[N, 2N]$, $\sum_N w_N(x) \equiv 1$ and $w_N^{(j)}(x) \ll_j N^{-j}$. We then write $S' = \sum_{\substack{c \equiv 0(M) \\ qM > c > 0}} \sum_N c^{-1} S_c(N)$. where

$$S_c(N) = \sum_{n=1}^{\infty} S(An, 1; c) \frac{1}{n^{1/2+\alpha}} J_{\kappa-1}\left(\frac{4\pi\sqrt{An}}{c}\right) w_N(n).$$

Proposition 4.9. *With S' and $\gamma = \max(0, Re(\alpha))$. as before,*

$$S' = q^{-\gamma+\epsilon} M^{-1+\epsilon} A^{\frac{1}{2}+\epsilon}.$$

This follows immediately from the previous lemma summing $S_c(N)$ over all dyadic N up to height $q^{1+\epsilon}$.

The $S_c(N)$ meet the criteria for Poisson summation, so we have

$$S_c(N) = \sum_{k \in \mathbb{Z}} a_A(k; c) r_A(k; c).$$

with

$$a_A(k; c) := \sum_{x(c)} \frac{1}{c} S(Ax, 1; c) e(\frac{kx}{c}),$$

and

$$r_A(k;c) := \int_0^\infty J_{\kappa-1}\left(\frac{4\pi\sqrt{Ax}}{c}\right) w_N(x) x^{-1/2-\alpha} e(\frac{kx}{c}) dx$$

We next need the following lemmas that bound the components of $S_c(N)$.

Lemma 4.10. *For integer A, k, c we have,*

$$|a_A(k;c)| \leq (A,c)$$

Proof. Opening the Kloosterman sum we have,

$$a_A(k;c) = \frac{1}{c}\sum_{x(c)}\sideset{}{^*}\sum_{y(c)} e\left(\frac{Axy+\bar{y}}{c}\right) e\left(\frac{kx}{c}\right).$$

The sum over x vanishes unless $Ay \equiv -k(c)$. in which case it gives a factor of c. This implies $(A,c)|(k,c)$, and so $\frac{A}{(A,c)}y \equiv -\frac{k}{(A,c)}\left(\frac{c}{(A,c)}\right)$. But then,

$$y \equiv -\overline{\left(\frac{A}{(A,c)}\right)}\frac{k}{(A,c)} \pmod{\left(\frac{c}{(A,c)}\right)}.$$

There are at most (A,c) such y We conclude $|a_A(k;c)| \leq (A,c)$. Note that if $k=0$ we are left with $\sum_{x(c)}\sideset{}{^*}\sum_{y(c)} e\left(\frac{Axy+\bar{y}}{c}\right)$ which vanishes since $Ay \neq 0(c)$, since y is a unit and A is coprime to c, because $A|L$, $L$is coprime to M, and $M|c$. This gives that $c \not| A$ or $c=1$. $\square$

We need a definition and Lemma from [9] to bound the stationary part of the Fourier transform. We say that a family of smooth functions on R_+^d, $\{\omega_T\}_{T\in\mathcal{F}}$ is X-inert for some $X \geq 0$ if for all $\mathbf{j} = \{j_1, \ldots, j_d\} \in Z_+^d$ we have,

$$|\omega_T^{\mathbf{j}}(x_1, \ldots, x_d| \ll_{\mathbf{j}} \frac{1}{|x_1|^{j_1}\cdots|x_d|^{j_d}} X^{j_1+\cdots+j_d}.$$

Lemma 4.11. *Suppose ω is a family of X-inert functions with compact support on $[Z,2Z]$, so that $\omega^{(j)}(t) \ll \left(\frac{Z}{X}\right)^{-j}$. Also, suppose $\phi^{(j)}(t) \ll \frac{Y}{Z^j}$ for $Y \gg X^2Q^\epsilon$, Let*

$$I = \int_{-\infty}^{\infty} \omega(t) e(\phi(t)).$$

If $\phi'(t) \gg \frac{Y}{Z}$ for all t in the support of ω, then for arbitrarily large $K > 0$ we have,

$$I \ll_K Q^{-K} \tag{6}$$

If $\phi''(t) \gg \frac{Y}{Z^2}$ for all t in the support of ω, and there exists a $t_0 \in \mathbb{R}$ such that $\phi'(t) = 0$ then

$$I = \frac{e(\phi(t_0))}{\sqrt{\phi''(t_0)}} F(t_0) + O(Q^{-K}). \tag{7}$$

where F is X-inert (depending on K) and K can be chose to be arbitrarily large.

This is Lemma 10.2 of [9].

Lemma 4.12. *For $T > 0$,*

$$r_A(k;c) \ll \begin{cases} N^{-Re(\alpha)} \frac{N\sqrt{A}}{c} q^{\epsilon} \left(1 + \frac{|k|N}{Tc}\right)^{-100} & c \gg T^{-1}\sqrt{Ax} \\ N^{-Re(\alpha)} \frac{c}{\sqrt{A}} q^{\epsilon} \left(1 + \frac{|k|\sqrt{N}}{\sqrt{A}}\right)^{-100} & c \ll T^{-1}\sqrt{Ax} \end{cases}$$

We take $T = q^{\epsilon}$

Proof. Case: $c \gg T^{-1}\sqrt{Ax}$. We will have $T = q^{\epsilon}$. In this range $J_{\kappa-1}\left(\frac{4\pi\sqrt{Ax}}{c}\right) \sim \frac{4\pi\sqrt{Ax}}{c}$, and $J_{\kappa-1}^{(j)}(y) \ll_{\kappa} \min(1, y^{-j})$ so that

$$r_A(k;c) = N^{-Re(\alpha)} \int_{x=0}^{\infty} h_{N,\alpha}(x) e\left(\frac{-kx}{c}\right) dx.$$

where $h^{(j)} \ll q^{\epsilon} \frac{\sqrt{AN}}{c} \left(\frac{T(1+|\alpha|)}{N}\right)^{j}$. Further, $h_{N,\alpha}(x)$ is $(1+|\alpha|)$ is X-inert. (6) along with our assumption that $|Im(\alpha)| < q^{\epsilon}$ gives the claimed bound.

Case: $c \ll T^{-1}\sqrt{Ax}$.
In this range, $J_{\kappa-1}(4\pi y) = y^{-\frac{1}{2}} \sum_{\pm} e(-2y) g_{\kappa,\pm}(4\pi y)$ where $g_{\kappa,\pm}^{(j)}(4\pi y) \ll_{\kappa} y^{-j}$. Taking $y = \frac{\sqrt{Ax}}{c}$, we have $r_A(k;c) = \frac{c^{\frac{1}{2}}}{(AN)^{\frac{1}{4}} N^{Re(\alpha)}} \int_{x=0}^{\infty} e(\pm 2\frac{\sqrt{Ax}}{c} - \frac{kx}{c}) w_{\kappa,\alpha}(x) dx$, where w satisfies the same type of derivative bounds as w. For $k \asymp \frac{\sqrt{A}}{\sqrt{x}} \asymp \frac{\sqrt{A}}{\sqrt{N}}$, we have the asymptotic from Lemma 4.11 (regardless of whether the stationary point is in the support of the weight function),

$$r_A(k;c) = N^{-Re(\alpha)} \frac{c}{\sqrt{A}} + O\left(q^{-100}\left(\frac{c}{|k|\sqrt{N}}\right)^2\right).$$

On the other hand, if k is not in this range, then the function is oscillatory and (6),

$$r_A(k;c) \ll \frac{c^{\frac{1}{2}}}{(AN)^{\frac{1}{4}} N^{\alpha}} \left((1+|\alpha|)\frac{\sqrt{AN}}{c} + \frac{|k|\sqrt{N}}{c}\right)^{-100}.$$

We unify these bounds (using our assumption $|\alpha| \ll q^{\epsilon}$) with

$$r_A(k;c) \ll N^{-Re(\alpha)} \frac{c}{\sqrt{A}} q^{\epsilon} \left(1 + \frac{|k|\sqrt{N}}{\sqrt{A}}\right)^{-100}.$$

□

Proof of Propostion 4.9. We now sum our estimates for $S_c(N)$ in the two ranges of c. This gives

$$\sum_{T^{-1}\sqrt{Ax} \leq c \leq C} c^{-1} S_c(N) \ll \sqrt{A} N^{-1} \sum_{c} c^{-2} \sum_{k \neq 0} (A,c) M^{\epsilon} \left(1 + \frac{|k|N}{c}\right)^{-100} \ll N^{-Re(\alpha)} M^{-1+\epsilon} A^{\frac{1}{2}+\epsilon} C^{\epsilon},$$

and

$$\sum_{c\le T^{-1}\sqrt{Ax}} c^{-1}S_c(N) \ll \sqrt{A}N^{-1}\sum_c c^{-1}\sum_{k\neq 0}(A,c)\frac{c}{\sqrt{A}}\left(1+\frac{|m|\sqrt{N}}{\sqrt{A}}\right)^{-100} \ll N^{-Re(\alpha)}M^{-1+\epsilon}A^{\frac{1}{2}+\epsilon}C^{\epsilon}.$$

It remains to check the case when $k=0$, in which case $a_A(k;c)$ vanishes unless $M=1$ and $c=1$. In this case, $a_A(k;c)=1$ and we are necessarily in the range when the Bessel function is oscillatory, so we have the bound $r_A(k;c) \ll N^{-Re(\alpha)}\frac{c}{\sqrt{A}}q^{\epsilon}$. It with $c=1$, it is clear this bound is sufficient. □

Plugging this estimate back in we are left with,

$$|\mathcal{E}_1| \ll q^{-1-\gamma+\epsilon}\sum_{LM=q}\sum_{\substack{\ell|L^\infty\\ \ell\le Y}}\frac{\ell}{\nu(\ell)^2}\sum_{d_1,d_2|\ell}c_\ell(d_1)c_\ell(d_2)\sum_{u|L}u\frac{\mu(u)}{\nu(u)}\sum_{a|u}\sum_{e_1|d_1}A^{\frac{1}{2}+\epsilon}.$$

Recall $A=\frac{d_1u(e_1,\frac{u}{a})}{ae_1}$. So,

$$|\mathcal{E}_1| \ll q^{-1-\gamma+\epsilon}Y^{\epsilon}\sum_{L|q}\sum_{\substack{\ell|L^\infty\\ \ell\le Y}}\frac{\ell}{\nu(\ell)^2}\sum_{d_1,d_2|\ell}c_\ell(d_1)c_\ell(d_2)(d_1)^{\frac{1}{2}} + O((qY)^{\epsilon}qY^{-2\gamma_0}).$$

This is bounded by $q^{-1-\gamma+\epsilon}$ via an application of Lemma 4.6.

5. Second Moment

Again, the primary tools for studying the second moment will be an approximate functional equation, as well as Corollary 4.2.

Lemma 5.1 (Approximate Functional Equation)**.** *For α,β shifts with real part less than $1/2$ in absolute value and imaginary part bounded by $O(q^{\epsilon})$, we have*

$$L(\frac{1}{2}+\alpha,f)L(\frac{1}{2}+\beta,f)=\sum_{m,n}\frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}+\alpha}n^{\frac{1}{2}+\beta}}V_{\alpha,\beta}(\frac{mn}{q})+\sum_{m,n}\frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}-\alpha}n^{\frac{1}{2}-\beta}}V_{-\alpha,-\beta}(\frac{mn}{q}) \quad (8)$$

where

$$V_{\alpha,\beta}(x):=\frac{1}{2\pi i}\int_{(2)}(x)^{-s}g_{\alpha,\beta}(s)\frac{G(s)ds}{s},$$

$$g_{\alpha,\beta}(s):=(2\pi)^{-2s}\frac{\Gamma(\alpha+s+\frac{\kappa}{2})\Gamma(\beta+s+\frac{\kappa}{2})}{\Gamma(\alpha+\frac{\kappa}{2})\Gamma(\beta+\frac{\kappa}{2})}.$$

and

$$X_{\alpha,\beta}:=\left(\frac{2\pi}{\sqrt{q}}\right)^{2(\alpha+\beta)}\frac{\Gamma(\alpha+\frac{\kappa}{2})\Gamma(\beta+\frac{\kappa}{2})}{\Gamma(-\alpha+\frac{\kappa}{2})\Gamma(-\beta+\frac{\kappa}{2})}.$$

with $G(s)$ an even, entire function of at most polynomial growth with $G(0)=1$.

Proof of 5.1. Consider,

$$J_{\alpha,\beta} := \frac{1}{2\pi i}\int_{(2)} \Lambda(\frac{1}{2}+\alpha+s)\Lambda(\frac{1}{2}+\beta+s)\frac{G(s)ds}{s},$$

where $G(s)$ is as above.

$$\Lambda(\frac{1}{2}+\alpha)\Lambda(\frac{1}{2}+\beta)+\frac{1}{2\pi i}\int_{(-2)} \Lambda(\frac{1}{2}+\alpha+s)\Lambda(\frac{1}{2}+\beta+s)\frac{G(s)ds}{s},$$

Making the change of variable $s \to -s$ we get

$$J_{\alpha,\beta} = \Lambda(\frac{1}{2}+\alpha)\Lambda(\frac{1}{2}+\beta) - J_{-\alpha,-\beta}.$$

Rearranging and plugging in the definition of the Λ

$$\left(\frac{\sqrt{q}}{2\pi}\right)^{1+\alpha+\beta}\Gamma(1/2+\alpha+\frac{k-1}{2})\Gamma(1/2+\beta+\frac{k-1}{2})L(1/2+\beta,f)L(1/2+\alpha,f) = J_{\alpha,\beta}+J_{-\alpha,-\beta}.$$

Dividing through,

$$\begin{aligned}L(\frac{1}{2}+\alpha,f)L(\frac{1}{2}+\beta,f) &= \frac{1}{2\pi i}\int_{(2)} q^s g_{\alpha,\beta}(s)L(\frac{1}{2}+\alpha+s)L(\frac{1}{2}+\beta+s)\frac{G(s)ds}{s}\\ &\quad + X_{\alpha,\beta}\frac{1}{2\pi i}\int_{(2)} q^s g_{-\alpha,-\beta}(s)L(\frac{1}{2}-\alpha+s)L(\frac{1}{2}-\beta+s)\frac{G(s)ds}{s}\end{aligned}$$

Define,

$$I_{\alpha,\beta} := \frac{1}{2\pi i}\int_{(2)} q^s g_{\alpha,\beta}(s)L(\frac{1}{2}+\alpha+s)L(\frac{1}{2}+\beta+s)\frac{G(s)ds}{s}$$

Then we have

$$L(\frac{1}{2}+\alpha,f)L(\frac{1}{2}+\beta,f) = I_{\alpha,\beta}+X_{\alpha,\beta}I_{-\alpha,-\beta}$$

We expand into the Dirichlet series giving,

$$I_{\alpha,\beta} = \frac{1}{2\pi i}\int_{(2)} g_{\alpha,\beta}(s)\sum_{m,n}\frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}+\alpha+s}n^{\frac{1}{2}+\beta+s}}.$$

Pulling the sums through,

$$L(\frac{1}{2}+\alpha,f)L(\frac{1}{2}+\beta,f) = \sum_{m,n}\frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}+\alpha}n^{\frac{1}{2}+\beta}}V_{\alpha,\beta}(\frac{mn}{q})+\sum_{m,n}\frac{\lambda_f(m)\lambda_f(n)}{m^{\frac{1}{2}-\alpha}n^{\frac{1}{2}-\beta}}V_{-\alpha,-\beta}(\frac{mn}{q}). \quad \square$$

From here forward we place the additional restriction on G that it has zeros at $\alpha+\beta$ and $-\alpha-\beta$. These zeros will cancel poles of the zeta function that will arise. As in the calculation of the first moment, we split into the main and error terms,

$$\sum_{f\in\mathcal{H}_k^*(q)} I_{\alpha,\beta} = \frac{1}{2\pi i}\int_{(2)} g_{\alpha,\beta}(s)\sum_{m,n}\frac{A_q(n,m)+\mathcal{E}_q(n,m)}{m^{\frac{1}{2}+\alpha+s}n^{\frac{1}{2}+\beta+s}} = A_{\alpha,\beta}+\mathcal{E}_{\alpha,\beta}.$$

Proposition 5.2. *For $\epsilon > 0$,*

$$A_{\alpha,\beta} = R + O(q^{-1/2-\frac{Re(\alpha)+Re(\beta)}{2}+\epsilon}), \tag{9}$$

where

$$R = \frac{\phi(q)}{q}\zeta(1+\alpha+\beta)\prod_{p|q}\frac{(1+p^{-2-\alpha-\beta})(1-p^{-1-\alpha-\beta})}{(1-p^{-2(1+\alpha)})(1-p^{-2(1+\beta)})}.$$

Proof. Via orthogonality,

$$\frac{1}{2\pi i}\int_{(2)} g_{\alpha,\beta}(s)\sum_{m,n}\frac{A_q(n,m)}{m^{\frac{1}{2}+\alpha+s}n^{\frac{1}{2}+\beta+s}}q^s\frac{G(s)ds}{s}$$
$$= \frac{1}{2\pi i}\frac{\phi(q)}{q}\int_{(2)} q^s g_{\alpha,\beta}(s)\zeta(1+\alpha+\beta+2s)\prod_{p|q}\frac{(1+p^{-2-\alpha-\beta-2s})(1-p^{-1-\alpha-\beta-2s})}{(1-p^{-(2(1+\alpha+s))})(1-p^{-(2(1+\beta+s))})}\frac{G(s)ds}{s}.$$

Shift contours to $(-1/2 - \frac{Re(\alpha)+Re(\beta)}{2} + \epsilon)$. Recall that $G(s)$ was selected to have zeros cancelling the zeta function. We pick up a residue at $s = 0$. For s on this line we have,

$$\Big|\prod_{p|q}\frac{(1+p^{-2-\alpha-\beta-2s})(1-p^{-1-\alpha-\beta-2s})}{(1-p^{-(2(1+\alpha+s))})(1-p^{-(2(1+\beta+s))})}\Big| < c(\epsilon_0)\prod_{p|q}2 \ll q^{\epsilon_0}.$$

The remaining integral is $O(q^{-1/2-\frac{Re(\alpha)+Re(\beta)}{2}+\epsilon})$, so we have

$$R = \frac{1}{2\pi i}\frac{\phi(q)}{q}\int_{(2)} g_{\alpha,\beta}(s)\zeta(1+\alpha+\beta+2s)\prod_{p|q}\frac{(1+p^{-2-\alpha-\beta-2s})(1-p^{-1-\alpha-\beta-2s})}{(1-p^{-(2(1+\alpha+s))})(1-p^{-(2(1+\beta+s))})}\frac{G(s)ds}{s} =$$
$$R + O(q^{-1/2-\frac{Re(\alpha)+Re(\beta)}{2}+\epsilon}),$$

as claimed. □

Putting this together, we get $I_{\alpha,\beta} = R + O(q^{-1/4-\min(Re(\alpha),Re(\beta))+\epsilon})$, from which Proposition5.2 follows.

6. Poisson Summation in Two Variables

As in the one variable case, we use Lemmas 4.5 and 4.6 to obtain,

$$\mathcal{E}_{\alpha,\beta} = 2\pi i^{-k}\sum_{m,n\geq 1}\frac{1}{m^{1/2+\alpha}n^{1/2+\beta}}\sum_{LM=q}\frac{\mu(L)}{\nu(L)}\sum_{\substack{\ell|L^\infty\\ \ell\leq Y}}\frac{\ell}{\nu(\ell)^2}\sum_{d_1,d_2|\ell}c_\ell(d_1)c_\ell(d_2)\sum_{\substack{u|(m,L)\\ v|(n,L)}}\frac{uv}{(u,v)}\frac{\mu(\frac{uv}{(u,v)^2})}{\nu(\frac{uv}{(u,v)^2})}\sum_{\substack{a|(\frac{m}{u},u)\\ b|(\frac{m}{v},v)}}$$
$$\sum_{e_1|(d_1,\frac{m}{a^2u})}\sum_{\substack{c\equiv 0(M)\\ c>0}}\frac{S(m',n';c)}{c}J_{\kappa-1}\big(\frac{4\pi\sqrt{m'n'}}{c}\big)V_{\frac{1}{2}+\alpha,\frac{1}{2}+\beta}\left(\frac{mn}{q}\right) + O((qY)^\epsilon qY^{-2\gamma_0}),$$

where $m' = \frac{md_1}{a^2e_1^2}$ and $n' = \frac{nd_2}{b^2e_2^2}$.

Applying the same elementary arguments as before and letting, $A = \frac{d_1 u(e_1, \frac{u}{a})}{ae_1}, B = \frac{d_2 v(e_2, \frac{v}{b})}{be_2}, D_1 = au\frac{e_1}{(e_1, \frac{e_1}{a(u,v)})}$ and $D_2 = bv\frac{e_2}{(e_2, \frac{e_2}{b(u,v)})}$ this becomes,

$$\mathcal{E}_{\alpha,\beta} = 2\pi i^{-k} \sum_{LM=q} \frac{\mu(L)}{\nu(L)} \sum_{\substack{\ell | L^\infty \\ \ell \leq Y}} \frac{\ell}{\nu(\ell)^2} \sum_{d_1, d_2 | \ell} c_\ell(d_1) c_\ell(d_2) \sum_{u|L} u \frac{\mu(u)}{\nu(u)} \sum_{a|u} \sum_{e_1 | d_1} S,$$

where

$$S := \sum_{\substack{c \equiv 0(M) \\ c>0}} \sum_{0<m,n} \frac{S(An, Bm; c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{ABmn}}{c}\right) V_{\frac{1}{2}+\alpha, \frac{1}{2}+\beta}\left(\frac{mnD_1D_2}{q}\right).$$

For $c > (qM) := C$, applying the Weil bound to the Kloosterman sums gives the desired bound.

Otherwise, we apply a partition of unity in both variables $w_{N_1,N_2}(x,y)$ so that w is supported on $[N_1, 2N_1] \times [N_2, 2N_2]$.

Applying Poisson summation gives,

$$S = \sum_{N_1, N_2 \geq 0} \sum_{\substack{c \equiv 0 \\ 0<c \leq C}} c^{-1} S_{N_1,N_2}(c),$$

with

$$S_c(N_1, N_2) := \sum_{r,t \in \mathbb{Z}} a_{A,B}(r,t;c) r_{A,B}(r,t;c).$$

where

$$a_{A,B}(r,t;c) := \frac{1}{c^2} \sum_{x_1, x_2(c)} S(Ar, Bt; c) e\left(\frac{rx_1}{c}\right) e\left(\frac{tx_2}{c}\right),$$

and

$$r_{A,B}(r,t;c) := \int_{x=0}^{\infty} \int_{y=0}^{\infty} \frac{1}{x^{\frac{1}{2}+\alpha} y^{\frac{1}{2}+\beta}} w_{N_1,N_2}(x,y) J_{\kappa-1}\left(\frac{4\pi\sqrt{ABxy}}{c}\right) e\left(-\frac{rx}{c} - \frac{ty}{c}\right) dxdy.$$

Lemma 6.1. *For A, B such that $c \not| A, B$ we have,*

$$|a_{A,B}(r,t;c)| = \begin{cases} (A,B,c) & \textit{if } rt \equiv AB(c), \\ 0 & \textit{Otherwise.} \end{cases}$$

Note that this implies if r or t equals 0 we are in the latter case, unless $c = 1$.

Proof. Opening the Kloosterman sum, we have

$$|a_{A,B}(r,t;c)| \ll \frac{1}{c^2} \sum_{x_1, x_2(c)} \sum_{y(c)}^{*} e\left(\frac{Ax_1 y + Bx_2 \overline{y}}{c}\right) e\left(\frac{rx_1}{c}\right) e\left(\frac{tx_2}{c}\right).$$

From this, the second condition can be read off, since the sum vanishes unless both $Ay \equiv -r(c)$ and $B\bar{y} \equiv -t(c)$. Supposing this condition is satisfied, we can then simplify via the triangle inequality to

$$|a_{A,B}(r,t;c)| \leq \sum_{\substack{y(c) \\ Ay\equiv -r(c) \\ B\bar{y}\equiv -r(c)}}^{*} 1.$$

The first condition implies,

$$y \equiv -\frac{A}{(A,c)}\frac{r}{(A,c)} \pmod{\frac{c}{(A,c)}},$$

and the second

$$y \equiv -\overline{\left(\frac{B}{(B,c)}\right)}\,\overline{\left(\frac{t}{(B,c)}\right)} \pmod{\frac{c}{(B,c)}},$$

this implies a unique congruency class for $y \pmod{\operatorname{lcm}\left(\frac{c}{(A,c)}, \frac{c}{(B,c)}\right)}$ i.e. $\pmod{\left(\frac{c}{(A,B,c)}\right)}$ completing the proof. $\square$

Lemma 6.2. *For $T > 0$, any fixed, small $\epsilon > 0$, for any $K > 0$,*

$$|r_{A,B}(r,t;c)| \ll_K \begin{cases} T\frac{N_1N_2\sqrt{AB}}{cN_1^{Re(\alpha)}N_2^{Re(\beta)}}\left(1+\frac{|t|N_1}{c}\right)^{-100}\left(1+\frac{|r|N_2}{c}\right)^{-100} & \frac{\sqrt{ABN_1N_2}}{c} \ll T, \\ \frac{c}{N_1^{Re(\alpha)}N_2^{Re(\beta)}\sqrt{AB}} + O((1+|\alpha|)T^{-K}) & \frac{\sqrt{ABN_1N_2}}{c} \gg T \ \& \ |AB-rt| \leq \frac{cr}{N_2}q^{\epsilon}, \\ (1+|\alpha|)(1+|\beta|)T^{-K} & \textit{otherwise}. \end{cases}$$

Taking $T = q^{\epsilon}(1+|\alpha|)^2(1+|\beta|)^2$ for some small $\epsilon > 0$ makes it clear that only the first term and the second main term will contribute, since when we sum the others we can still bound them by an arbitrary power of the level.

Proof of 6.2. For $\frac{\sqrt{ABN_1N_2}}{c} \ll T$, we have that $J_{\kappa-1}(y) \sim y$ so that we may write

$$r_{A,B}(r,t;c) = \frac{\sqrt{AB}}{cN_1^{Re(\alpha)}N_2^{Re(\beta)}}\int_{x=0}^{\infty}\int_{y=0}^{\infty} v_{N_1,N_2,\alpha,\beta}(x,y)e\left(\frac{-rx-ty}{c}\right)dxdy,$$

where

$$\frac{\partial^{(j)}}{\partial x^j}v_{N_1,N_2,\alpha}(x,y) \ll_j (1+|\alpha|)N_1^{-j} \text{ and } \frac{\partial^{(j)}}{\partial y^j}v_{N_1,N_2,\alpha,\beta}(x,y) \ll_j (1+|\beta|)N_2^{-j}.$$

From here (6) noting v is $(1+|\alpha)(1+|\beta|)$-inert gives the claimed bound.

For $\frac{\sqrt{ABN_1N_2}}{c} \gg T$, we have $J_{\kappa-1}(4\pi z) = \sum_{\pm} e(2z)g_{k,\pm}(2z)$, with $g_{\kappa,\pm}^{(j)}(2z) \ll_{\kappa} \min(1, z^{-j})$. Then

$$r_{A,B}(r,t;c) = \frac{\sqrt{c}}{N_1^{\frac{3}{4}+Re(\alpha)}N_2^{\frac{3}{4}+Re(\beta)}(AB)^{\frac{1}{4}}}\int_{x=0}^{\infty}\int_{y=0}^{\infty} h_{k,N_1,N_2,\alpha,\beta}(x,y)e\left(\frac{2\sqrt{ABxy}}{c}-\frac{rx}{c}-\frac{ty}{c}\right)dxdy.$$

If there is no cancellation between phases, we see that by (6),

$$r_{A,B}(r,t;c) \ll N_1^{-Re(\alpha)} N_2^{-Re(\beta)} T((1+|\alpha|)(1+|\beta|))^K \left(1 + \frac{\sqrt{ABN_1N_2}}{c}\right)^{-K} \left(1 + \frac{|r|N_2}{c}\right)^{-2} \left(1 + \frac{|t|N_1}{c}\right)^{-2}.$$

Otherwise, we apply Lemma 4.11 in x with $X = (1+|\alpha|)$ and $Q = \left(1 + \frac{\sqrt{ABN_1N_2}}{c(1+|\alpha|)^2(1+|\beta|)^2}\right)$. Note that $Q \gg (1+|\alpha|)T$.

$$r_{A,B}(r,t;c) = \frac{c}{N_1^{Re(\alpha)} N_2^{1+Re(\beta)} \sqrt{AB}} \int_0^\infty h_{k,N_2,\beta}(y) e\left(\frac{yAB}{rc} - \frac{ty}{c}\right) dy + O((1+|\alpha|)^{2K}(1+|\beta|)^{2K} T^{-K}).$$

If,

$$\frac{|AB - rt|N_1}{ct} < T,$$

we bound the integral trivially. Otherwise, we integrate by parts giving

$$r_{A,B}(r,t;c) \ll \frac{c}{N_1^{Re(\alpha)} N_2^{Re(\beta)} \sqrt{AB}} \left(1 + \frac{|r|N_2}{c}\right)^{-2} \left(1 + \frac{|t|N_1}{c}\right)^{-2} ((1+|\alpha|)(1+|\beta|))^K T^{-K}.$$

□

Proposition 6.3. *Let $\gamma = \min(Re(\alpha), Re(\beta))$. Then,*

$$\Big| \sum_{\substack{N_1N_2 < q^{1+\epsilon} \\ N_1,N_2 \ dyadic}} \sum_{\substack{c \equiv 0(M) \\ c \ll q^\epsilon \sqrt{ABN_1N_2}}} \frac{1}{c} s_{N_1,N_2}(c) \Big| \ll q^{-\gamma+\epsilon} M^{-1+\epsilon} (AB)^{\frac{1}{2}}.$$

Proof. We have,

$$\begin{aligned} &\Big| \sum_{\substack{N_1N_2 < q^{1+\epsilon} \\ N_1,N_2 \text{ dyadic}}} \sum_{\substack{c \equiv 0(M) \\ c \ll q^\epsilon \sqrt{ABN_1N_2}}} \frac{1}{c} s_{N_1,N_2}(c) \Big| \\ &\qquad \ll \frac{q^\epsilon}{\sqrt{AB}} \Big| \sum_{\substack{N_1N_2 < q^{1+\epsilon} \\ N_1,N_2 \text{ dyadic}}} \frac{1}{N_1^\alpha N_2^\beta} \sum_{\substack{c \equiv 0(M) \\ c \ll q^\epsilon \sqrt{ABN_1N_2}}} (A,B,c) \sum_{\substack{r,t \neq 0 \\ |N_1N_2 - rt| \leq \frac{ct}{N_1} q^\epsilon \\ rt \equiv AB(c)}} 1 \\ &\qquad\qquad \ll q^\epsilon (AB)^{-\frac{1}{2}+\epsilon} \Big| \sum_{\substack{N_1N_2 < q^{1+\epsilon} \\ N_1,N_2 \text{ dyadic}}} \frac{1}{N_1^\alpha N_2^\beta} \left(\sqrt{ABN_1N_2} + AB\right). \end{aligned}$$

The bound follows taking $U = N_1N_2$ and bounding the number of such U by the divisor function.

It remains to consider the cases when $r = 0, t = 0$, and $r = t = 0$. We will show that the contribution from these cases is bounded by q^ϵ. In all of these cases, the sum is 0 unless $c = 1$, which implies $M = 1$. In any of these case, we have that $a_{A,B}(r,t) = 1$.

If $\sqrt{ABN_1N_2} \ll q^{\epsilon}$, then we have via Lemma 6.2, that $r_{A,B}(r,t;1) \ll q^{\epsilon}(1+|r|)^{-2}(1+|t|)^{-2}$. In the case when one is nonzero, the sum converges when we sum over the other. We may then sum over $N_1N_2 < q^{\epsilon}$, to complete this case.

Otherwise, if $\sqrt{ABN_1N_2} \gg q^{\epsilon}$, by Lemma 6.2 we have that $r_{A,B}(r,t;1) \ll (1+|\alpha|)(1+|\beta|)T^{-K}$, so we may bound this term by q^{-100}, and after summing up over $N_1N_2 \leq q^{1+\epsilon}$, this term is still significantly small than q^{ϵ}. ☐

Proposition 6.4. *Let $\gamma = \min(Re(\alpha), Re(\beta))$. Then,*

$$\sum_{\substack{N_1N_2<q^{1+\epsilon}\\ N_1,N_2 \text{ dyadic}}} \sum_{\substack{c\equiv 0(M)\\ C\geq c\gg q^{\epsilon}\sqrt{ABN_1N_2}}} \frac{1}{c} s_{N_1,N_2}(c) \ll q^{1/2-\gamma+\epsilon}\epsilon C^{\epsilon} M^{-1+\epsilon}(AB)^{\frac{1}{2}}$$

Proof.

$$\Big| \sum_{\substack{N_1N_2<q^{1+\epsilon}\\ N_1,N_2 \text{ dyadic}}} \sum_{\substack{c\equiv 0(M)\\ C\geq c\gg q^{\epsilon}\sqrt{ABN_1N_2}}} \frac{1}{c} s_{M,N}(c)\Big| \ll q^{\epsilon}\sqrt{AB}\Big| \sum_{\substack{N_1N_2<q^{1+\epsilon}\\ N_1,N_2 \text{ dyadic}}} \frac{N_1N_2}{cN_1^{\alpha}N_2^{\beta}} \sum_{\substack{c\equiv 0(M)\\ C\geq c\gg q^{\epsilon}\sqrt{ABN_1N_2}}} \frac{(A,c)}{c^2} \sum_{\substack{r,t\neq 0\\ |r|\leq \frac{c}{N_2}\\ |t|\leq\frac{c}{N_1}\\ rt\equiv AB(c)}} 1\Big|$$

$$\ll q^{\epsilon}\sqrt{AB}\Big| \sum_{\substack{N_1N_2<q^{1+\epsilon}\\ N_1,N_2 \text{ dyadic}}} \frac{N_1N_2}{N_1^{Re(\alpha)}N_2^{Re(\beta)}} \sum_{\substack{c\equiv 0(M)\\ qM\geq c\gg q^{\epsilon}\sqrt{ABN_1N_2}}} \frac{(A,c)}{c^2}\left(1+\frac{c}{N_1N_2}\right)\Big|$$

$$\ll q^{\epsilon}M^{-1+\epsilon}\sqrt{AB} \sum_{\substack{N_1N_2<q^{1+\epsilon}\\ N_1,N_2 \text{ dyadic}}} \frac{\sqrt{N_1N_2}}{N_1^{Re(\alpha)}N_2^{Re(\beta)}}$$

The bound follows taking $U = N_1N_2$ and bounding the number of such U by the divisor function. ☐

Using this estimate and lemma 4.6 gives the desired result.

This completes the proof of Theorem 1.3 taking account of the term $X_{\alpha,\beta}I_{-\alpha,-\beta}$ in the functional equation

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