On the identification of k-inductively pierced codes using toric ideals

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Abstract

Neural codes are binary codes in $\{0, 1\}^n$; here we focus on the ones which represent the firing patterns of a type of neurons called place cells. There is much interest in determining which neural codes can be realized by a collection of convex sets. However, drawing these convex sets, particularly as the number of neurons in a code increases, can be very difficult. It has been shown that an algorithm for drawing Euler diagrams can be used to draw a class of codes that are said to be k-inductively pierced for $k = 0, 1, 2$. We use the toric ideal, to show sufficient conditions for a code to be 1- or 2-inductively pierced.

1 Introduction

John O'Keefe's discovery of place cells earned him the 2014 Nobel Prize for Physiology or Medicine [12]. Place cells are a type of neuron found in certain mammals (including rats, cats, and bats) that help them to locate themselves spatially by firing only when the mammal is in a certain part of its environment. The receptive fields, or areas where the neurons fire, are approximately convex, and can be represented by neural codes. Much study has been done on convex neural codes, specifically focusing on which codes are convex realizable, that is, for which codes one can draw convex receptive fields that correspond to the code [2, 3, 5, 6, 9, 10, 13]. However, if a convex realization exists it can be very difficult to actually draw one. Far less work has been done on how to draw realizations of neural codes. However, [8] have shown that the algorithm created by [14] to draw Euler diagrams can be used to create realizations of codes that are 0-, 1-, or 2-inductively pierced. In this paper, we explore sufficient conditions for 1- and 2-inductively pierced codes. In Section 2, we discuss the definitions necessary for our main results, which we present in Section 3. We conclude with a discussion of our findings in Section 4.

2 Background

We follow the definitions from [8].

Definition 2.1. A *neural code* on n neurons is a set of binary strings $\mathcal{C} \subseteq \{0, 1\}^n$. An element σ of $\mathcal{C}$ is a *codeword*.

We will assume that the empty codeword is always in a neural code.

Definition 2.2. A *realization* of a code $\mathcal{C}$ is a collection of sets $\mathcal{U} = \{U_1, \ldots, U_n\}$ where $U_i \in \mathbb{R}^d$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U}) := \{\sigma \in [n] \mid U_\sigma \setminus \bigcup_{j \in [n] \setminus \sigma} U_j \neq \emptyset\}$. A $U_i \in \mathcal{U}$ is a *place field* of the neuron i. A *zone* in $\mathcal{C}$ is the intersection of a collection of place fields $\{U_i\}$ and each zone corresponds to a codeword.

Definition 2.3. A code $\mathcal{C}$ is *convex open* if it is realizable by $\mathcal{U} = \{U_1, \ldots, U_n\}$ where all the U_i are convex open sets.

In this paper, we will work under with neural codes that are convex open and realizable in dimension 2. Furthermore, we will assume that our codes are well-formed.

Definition 2.4. A code $\mathcal{C}$ is *well-formed* if there exists realization of $\mathcal{C}$ such that

- The boundary curves of place fields intersect at only a finite number of points.
- At any given point, at most two boundaries of place fields intersect.
- Each zone is connected.

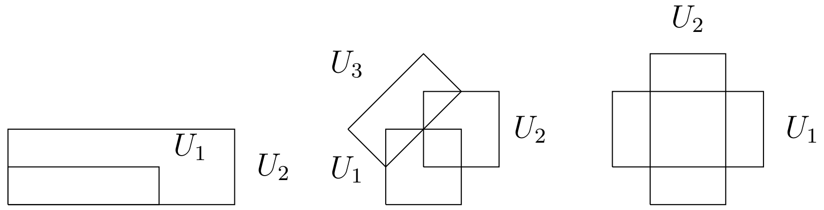

Figure 1: Figures that violate the three parts of the definition of well-formed

Example 2.5. As an example, consider the code $\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$. This neural code is convex open and realizable in dimension 2. A well-formed realization of $\mathcal{C}$ is shown in Figure 2. The realization is comprised of 6 zones, one for each codeword, including the empty zone.

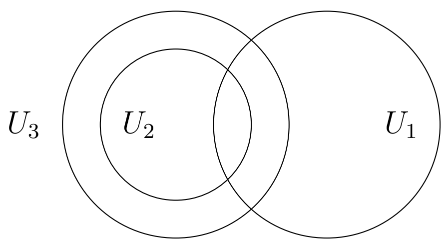

Figure 2: A realization of the code $\mathcal{C} = \{000, 100, 001, 101, 011, 111\}$

Definition 2.6. A *k-piercing* is a place field U_ℓ whose boundary intersects the boundaries of k other place fields $U_1, \ldots, U_k$ and adds exactly 2^k zones to an existing diagram. A k-piercing is identified by a zone in the diagram in which the place field U_ℓ is contained.

For example, in Figure 2, U_2 is a 1-piercing of U_1 identified by the zone 001. Notice that U_1 is not a 2-piercing; its boundary intersects the boundaries of two place fields, adding it does not add 4 new regions to the diagram.

Definition 2.7. Let $\mathcal{C} = \{\sigma_1, \ldots, \sigma_m\}$ be a code on n neurons. $\mathcal{C}$ is k-inductively pierced if there exists a k-piercing U_λ in $\mathcal{C}$ such that $\mathcal{C} \setminus \lambda := \{\hat{\sigma}_1, \ldots, \hat{\sigma}_m\}$ is k-inductively pierced where $\hat{\sigma}_i = \sigma_i$ except $\hat{\sigma}_i$ has a 0 in the λth position.

In this paper will will assume that if a code is k-inductively pierced, the code is not $(k+1)$-inductively pierced.

Definition 2.8. Define $\phi_c : \mathbb{F}_2[p_c \mid c \in \mathcal{C}] \to \mathbb{F}_2[x_i \mid i \in [n]]$

$$p_c \mapsto \prod_{i \in \mathrm{supp}(c)} x_i.$$

Then, $I_{\mathcal{C}} := \ker \phi_{\mathcal{C}}$ is the *toric ideal* of $\mathcal{C}$.

The toric ideal has the computational benefit of being relatively quick to compute using `Macaulay2` [7] with the `4ti2` package [1], or with `SAGE` [4].

In [8], the authors investigated algebraic signatures for 0-,1-, and 2-inductively pierced codes by considering generators of $I_{\mathcal{C}}$. In particular, they give necessary and sufficient conditions for 0-inductively pierced codes, along with necessary conditions for 1-inductively pierced codes.

Theorem 2.9 ([8])**.** *Let $\mathcal{C}$ be a well-formed neural code on n neurons.*

1. *The neural code $\mathcal{C}$ is 0-inductively pierced if and only if $I_{\mathcal{C}} = \langle 0 \rangle$.*

2. *If the neural code C is 0- or 1- inductively pierced then $I_{\mathcal{C}} = \langle 0 \rangle$ or generated by quadratics.*

3. *If the neural code C contains a triple intersection where the intersections are general, $I_{\mathcal{C}}$ contains a binomial of degree 3 of particular form, in particular $p_{111w}p_{000v}^2 - p_{100v}p_{010v}p_{001w}$ or $p_{111w} - p_{100\ldots0}p_{010\ldots0}p_{001w}$ where $v, w \in \{0,1\}^{n-3}$ correspond to zones in $\mathcal{C}(\mathcal{U})$.*

We note that the third claim in Theorem 2.9 includes the case of codes with 2-piercings (Proposition 4.3.1 in [11]) and the case of 2-inductively pierced codes, as summarized by Lemma 2.10.

Lemma 2.10. *If a neural code $\mathcal{C}$ contains a 2-piercing or is 2-inductively pierced, then $I_{\mathcal{C}}$ contains a cubic of form $p_{111w}p_{000v}^2 - p_{100v}p_{010v}p_{001w}$ or $p_{111w} - p_{100\ldots0}p_{010\ldots0}p_{001w}$ where $v = w$.*

We refer to the binomial $p_{111w}p_{000v}^2 - p_{100v}p_{010v}p_{001w}$ for $w, v \in \{0,1\}^{n-3}$ as a *cubic of a particular form* or *particular cubic.*

In the next section, we investigate sufficient conditions for 1- and 2-inductively pierced codes, and improve upon the second and third statements from Theorem 2.9.

3 Main Results

Throughout the rest of this paper, we assume that $\mathcal{C} = \mathcal{C}(\mathcal{U})$ is a convex open neural code on n neurons that is realizable in dimension 2, so $\mathcal{U} = \{U_1, \ldots, U_n\}$ and the $U_i \subset \mathbb{R}^2$.

Proposition 3.1. *Let $\mathcal{C}$ be a well-formed code on n neurons, and let $I_{\mathcal{C}}$ be its toric ideal. If there exists a cubic generator of $I_{\mathcal{C}}$ of the form $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v}$ (where $w, v \in \{0,1\}^{n-3}$), then $\mathcal{C}\backslash\{4,\ldots,n\}$. is 2-inductively pierced.*

Proof. The existence of such a cubic in $I_{\mathcal{C}}$ implies that the code contains a triple intersection of $U_1, U_2,$ and U_3 along with the associated singletons. Since the code is well formed, it must also contain the pairwise intersections. Thus, if all other neurons are removed, the remaining code is clearly 2-pierced. □

Corollary 3.2. *If $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$, then $\mathcal{C}$ is not 1-inductively pierced.*

Proof. Assume $\mathcal{C}$ is 1-inductively pierced. By Proposition 3.1, $\mathcal{C}\backslash\{4,\ldots,n\}$ contains a 2-piercing. Thus, U_1 cannot be a 0- or 1-piercing unless either U_2 or U_3 is removed. Since the same is true of U_2 and U_3, none of them can possibly be 0- or 1-piercings. Thus $\mathcal{C}$ is not 1-inductively pierced, since as place fields that are 0- or 1-piercings are removed, eventually $\mathcal{C}\backslash\{4,\ldots,n\}$ must be all that is left. However, $\mathcal{C}\backslash\{4,\ldots,n\}$ is not 1-inductively pierced, resulting in a contradiction. □

Corollary 3.3. *If $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$ and $w \neq v$, then $\mathcal{C}$ does not contain a 2-piercing, or the 2-piercing is contained within another receptive field and there must exist another cubic with $v = w$.*

Proof. If $w \neq v$, then there exists some zone in the potential 2-piercing that is contained in a different zone than other zones in the 2-piercing. Assume that the other zone does not contain an entire receptive field. Then, since the realization is well-formed, $\mathcal{C}$ must not contain a 2-piercing. If the other zone does contain an entire receptive field, then the 2-piercing must take place inside this other zone, thus there must also exist a cubic such that $v = w$. □

Lemma 3.4. *Let $\mathcal{C}$ be a code on n neurons such that in $\mathcal{C} \setminus \{3,\ldots,n\}$, U_1 is a 1-piercing of U_2 and there do not exist $i,j,k \in [n]$ such that in $\mathcal{C} \setminus ([n] \setminus \{i,j,k\})$, U_i is a 2-piercing of U_j and U_k. Then $\mathcal{C}$ is 1-inductively pierced.*

Proof. Suppose towards contradiction that $\mathcal{C}$ is not 1-inductively pierced. Then, there must be some neuron s such that U_s intersects both U_1 and U_2 such that fewer than 4 zones are created and neither U_1 nor U_2 is a 2-piercing of U_s. However, this can only happen if $\mathcal{C}$ is not well-formed. However, we assumed that $\mathcal{C}$ is well-formed, resulting in a contradiction. □

Theorem 3.5. *Let $\mathcal{C}$ be a well-formed code on n neurons, and let $I_{\mathcal{C}}$ be its toric ideal. If there exists a cubic of the form $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$ and in all such cases $w = v$, then $\mathcal{C}$ is 2-inductively pierced.*

Proof. We will proceed by proving the contrapositive. That is, we will assume that $\mathcal{C}$ is not 2-inductively pierced and show that whenever the toric ideal contains cubics of the particular form, there exists one such cubic with $w \neq v$.

By Proposition 3.1 $\mathcal{C}\backslash\{4,\ldots,n\}$ is 2-inductively pierced, so without loss of generality, U_1, U_2, U_3 are all two-piercings of each other.

Suppose that there exists some U_j that obstructs U_1, U_2, and U_3 from being 2-piercings of each other in $\mathcal{C}$. If U_j contains the triple intersection, then all zones other than the triple intersection are partially contained in U_j.

If U_j does not contain the triple intersection, then U_j partially contains it. In either case, we have the zones necessary to construct a cubic with $v \neq w$, as illustrated in Figure 3.

Furthermore, using the proof of Lemma 3.4, the only way for $\mathcal{C}$ to fail to be 2-inductively pierced is if there exists such a U_j. That is to say, there must be a curve U_j that obstructs U_1, U_2, and U_3 from all being two piercings of each other, so U_j cannot be a 0- or 1-piercing.

Thus, there must exist a cubic of form $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$ where $v \neq w$. □

Proposition 3.6. *Let $\mathcal{C}$ be a well formed code on n neurons. If there is no cubic of the form $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$ and $I_{\mathcal{C}} \neq \langle 0 \rangle$, then $\mathcal{C}$ is 1-inductively pierced.*

Proof. The lack of a cubic of form $p_{111w}p_{000v}^2 - p_{100w}p_{010v}p_{001v} \in I_{\mathcal{C}}$ implies that there do not exist $i, j, k \in [n]$ such that in $\mathcal{C} \setminus ([n] \setminus \{i, j, k\})$, U_i is a 2-piercing of U_j and U_k. Since $I_{\mathcal{C}} \neq \langle 0 \rangle$, $\mathcal{C}$ cannot be 0-inductively pierced, thus there must be some curves that intersect. Thus by Lemma 3.4 $\mathcal{C}$ must be 1-inductively pierced. □

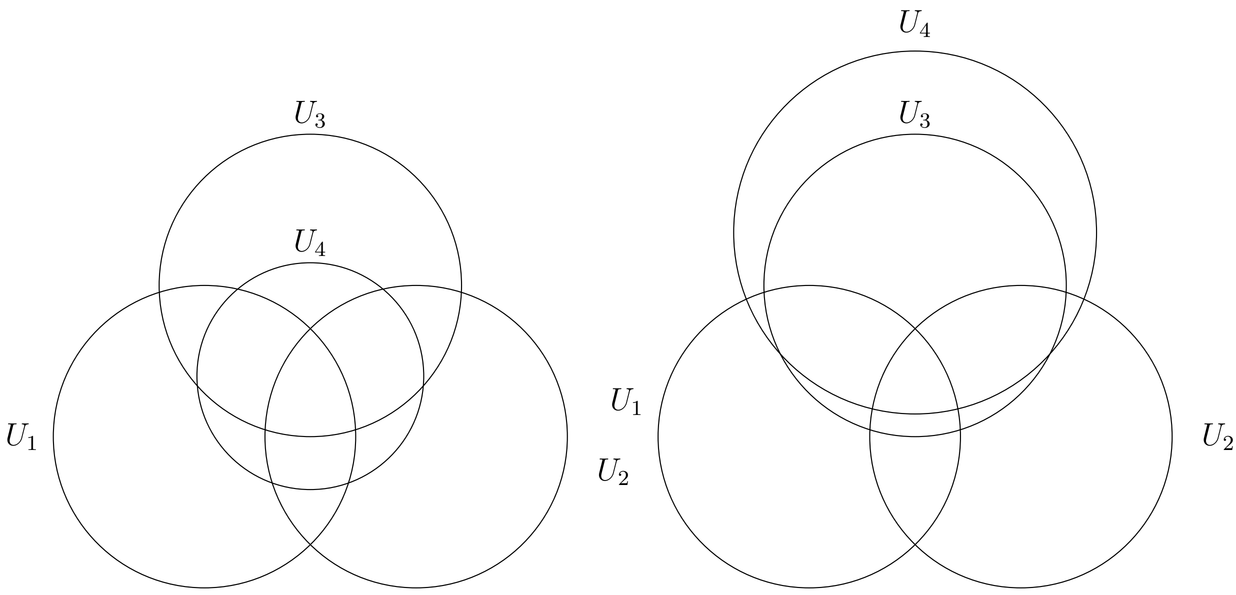

Figure 3: Examples of codes that are not 2-inductively pierced and have cubics of particular form where $v \neq w$ in $I_{\mathcal{C}}$. Note that on the left, U_4 contains the entire triple intersection $U_1 \cap U_2 \cap U_3$, and on the right U_4 partially contains the triple intersection.

The existence of particular cubic binomials in the toric ideal can tell us much about the code. However, computing the entire toric ideal is computationally expensive; we would much rather only look at generating sets. Unfortunately, the existence of particular cubics in the toric ideal does not imply they are generators.

Example 3.7. As discussed in [11], one generating set for the toric ideal of the code A1 $= \{000, 100, 010, 001, 110, 101, 011, 111\}$ is $\{p_{111} - p_{100}p_{010}p_{001},\ p_{110} - p_{100}p_{010},\ p_{101} - p_{100}p_{001},\ p_{011} - p_{010}p_{001}\}$. This generating set has a particular cubic.

Another generating set for for I_{A1}, which consists only of quadratics is $\{p_{110} - p_{100}p_{010},\ p_{101} - p_{100}p_{001},\ p_{011} - p_{010}p_{001},\ p_{111} - p_{110}p_{001}\}$.

From the second generating set for I_{A1} in Example 3.7, we notice that a priori we may not immediately see the existence of a particular cubic from the generators directly. However, there is a class of quadratics that can imply the existence of the particular cubics.

Definition 3.8. Let $\mathcal{C}$ be code on n neurons and $w, v \in \{0,1\}^{n-3}$. A pair of quadratics of a particular form

$$\begin{cases} p_{111w}p_{000v} - p_{110w}p_{001v} \\ p_{110w}p_{000v} - p_{100w}p_{010v} \end{cases} \text{or} \begin{cases} p_{111w}p_{000v} - p_{101w}p_{010v} \\ p_{101w}p_{000v} - p_{100w}p_{001v} \end{cases} \text{or} \begin{cases} p_{111w}p_{000v} - p_{011w}p_{100v} \\ p_{011w}p_{000v} - p_{010w}p_{001v} \end{cases}$$

are called *friendly quadratics.*

Proposition 3.9. *Let $z \in \{0,1\}^{n-3}$. If any of the friendly quadratics are in the generating set for the toric ideal $I_{\mathcal{C}}$ of $\mathcal{C}$, then $I_{\mathcal{C}}$ contains a cubic of the form $p_{111z}p_{000z}^2 - p_{100z}p_{010z}p_{001z}$.*

Proof. This is straightforward to see since $I_{\mathcal{C}}$ is an ideal, hence is closed under multiplication by elements of the ring. Consider the first case. Since $p_{111w}p_{000v} - p_{110z}p_{001z} \in I_{\mathcal{C}}$, we get that $p_{000z}, p_{001z} \in \mathbb{F}_2[p_c \mid c \in \mathcal{C}]$ and so,

$$p_{111z}p_{000z}^2 - p_{100z}p_{010z}p_{001z} = p_{000z}(p_{111z}p_{000z} - p_{110z}p_{001z}) \\ + p_{001z}(p_{110z}p_{000z} - p_{100z}p_{010z}) \in I_{\mathcal{C}}.$$

□

Example 3.10. Consider the following neural code on 5 neurons, $\mathcal{C}$ ={00001, 10001, 01001, 00011, 11001, 10011, 01011, 00111, 11011, 10111, 01111, 11111}. One generating set for $I_{\mathcal{C}}$ is

$$\begin{aligned} \langle &p_{00111}p_{11111} - p_{10111}p_{01111},\ p_{01011}p_{11111} - p_{11011}p_{01111},\ p_{10011}p_{11111} - p_{11011}p_{10111}, \\ &p_{00011}p_{11011} - p_{10011}p_{01011},\ p_{00011}p_{10111} - p_{10011}p_{00111},\ p_{00011}p_{11111} - p_{10011}p_{01111}, \\ &p_{00011}p_{01111} - p_{01011}p_{00111},\ p_{00011}p_{11111} - p_{01011}p_{10111},\ p_{00011}p_{11111} - p_{00111}p_{11011}, \\ &p_{01001}p_{10011} - p_{00011}p_{11001},\ p_{01001}p_{11011} - p_{11001}p_{01011},\ p_{01001}p_{10111} - p_{11001}p_{00111}, \\ &p_{01001}p_{11111} - p_{11001}p_{01111},\ p_{10001}p_{01011} - p_{00011}p_{11001},\ p_{10001}p_{11011} - p_{11001}p_{10011}, \\ &p_{10001}p_{01111} - p_{11001}p_{00111},\ p_{10001}p_{11111} - p_{11001}p_{10111},\ p_{00001}p_{11001} - p_{10001}p_{01001}, \\ &p_{00001}p_{10011} - p_{10001}p_{00011},\ p_{00001}p_{10111} - p_{10001}p_{00111},\ p_{00001}p_{01011} - p_{01001}p_{00011}, \\ &p_{00001}p_{01111} - p_{01001}p_{00111},\ p_{00001}p_{11011} - p_{00011}p_{11001},\ p_{00001}p_{11111} - p_{11001}p_{00111}\rangle. \end{aligned}$$

Note that no cubics appear in this generating set. However, $p_{00001}p_{11111} - p_{11001}p_{00111}$ and $p_{00001}p_{11001} - p_{10001}p_{01001}$ do. These are a pair of friendly quadratics, thus a cubic of particular form must appear in $I_{\mathcal{C}}$, namely a particular cubic with $w = v$. We see from a realization of $\mathcal{C}$ in Figure 4 that this code is 2-inductively pierced, even though it does not satisfy the conditions of Theorem 3.5.

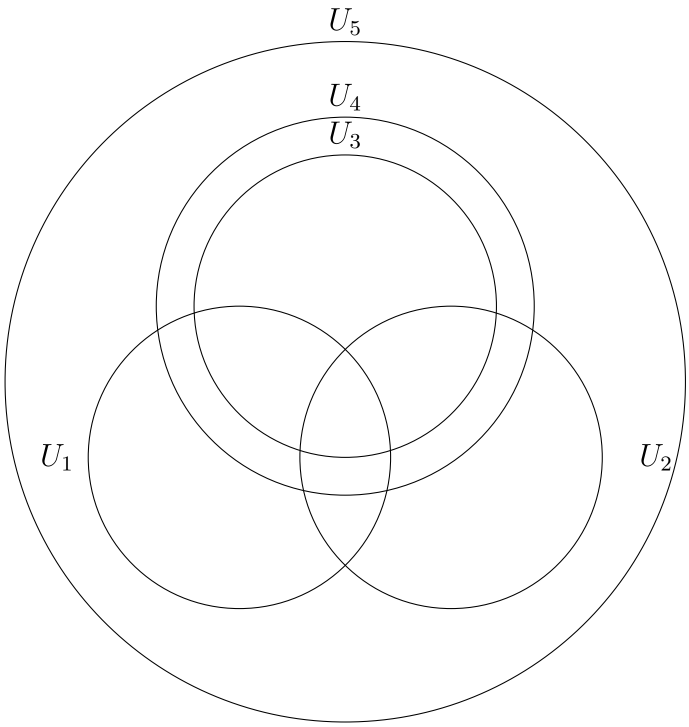

Figure 4: A realization of the code from Example 3.10.

4 Discussion

We have shown sufficient conditions for both 1- and 2-inductively pierced codes. However, these conditions possess significant weaknesses. Our sufficient condition for a code to be 1-inductively pierced relies on being able to analyze the entire toric ideal, a task that becomes unfeasible as for codes on N neurons, for large N. However, analyzing generating sets of the toric ideal is much easier. If we could classify all possible ways that a cubic of particular form can be generated, we could identify whether a cubic of particular form is in the toric ideal simply by looking an arbitrary generating set. If this were accomplished, Theorem 3.5 and Proposition 3.6 would become much more powerful.

Another direction for future study lies in identify which receptive fields could potentially form a 2-piercing. In this paper we assumed that we knew which three place fields were involved in a 2-piercing just from looking at the code. One could simply check every possible combination of three neurons, but as the number of neurons increases, this becomes far more difficult.

Further research in identifying which codes are well-formed and realizable in two dimensions would also be of great benefit. We have made the assumption that the codes we work with are well-formed and realizable in two dimensions, but as of yet we are unaware of the existence of any tools to determine these conditions.

Figure 5: A wombat, which may or may not have place cells, though linguistic analysis suggests the affirmative.

Acknowledgements

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