A Faster Randomized Algorithm for Counting Roots in $\mathbb{Z}/(p^k)$

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Applications

- **Exponential Sums:**

$$S(f,n) = \left| \sum_{i=1}^{n} e^{\frac{2\pi\sqrt{-1}g(x)}{n}} \right|$$

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- Work done by Cochrane and Zheng (2001) shows that estimating $S(f,n)$ is closely related to counting roots of $g \in \mathbb{Z}/(p^k)$.

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- Counting roots in $\mathbb{Z}/(p^k)$ is also closely related to counting roots in $\mathbb{Q}_p$.

Main Results

- There is a Las Vegas randomized algorithm that counts the roots of any $f \in \mathbb{Z}[x]$ (such that f is not identically 0 modulo p) of degree d in time:

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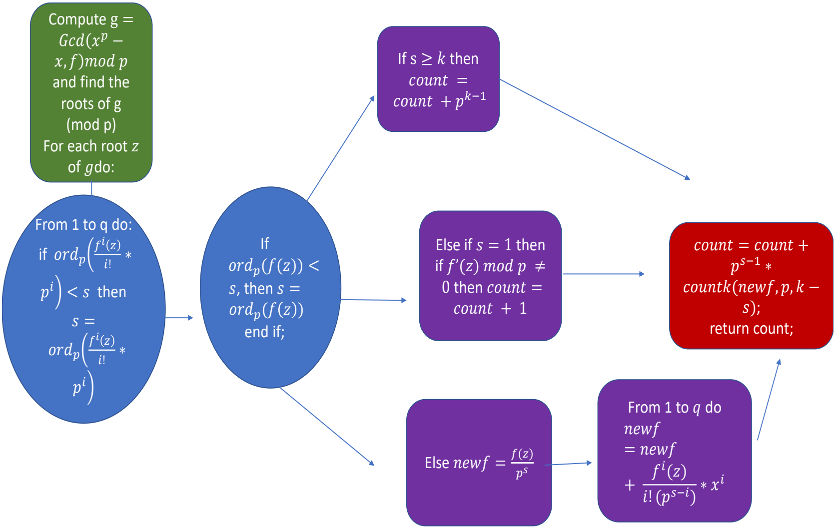
- Las Vegas algorithms are accepted by and occur frequently in algorithmic number theory (e.g., factoring polynomials over finite fields and primality testing).

The Algorithm



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- Then, given any root $\zeta \in \mathbb{Z}/(p)$ of f and an $\varepsilon \in \{0, 1, \cdots, p^{k-1} - 1\}$, we have a perturbation as follows (using a Taylor series expansion):

$$f(\zeta + p \cdot \varepsilon) = f(\zeta) + f'(\zeta)p\varepsilon + \frac{1}{2}f''(\zeta)p^2\varepsilon^2 + \ldots$$

$$\ldots + \frac{1}{(k-1)!}f^{(k-1)}(\zeta)p^{k-1}\varepsilon^{k-1} \mod p^k$$

The Key Trick

- By finding a maximum $s \in \{1, ..., k\}$ such that $p^s \mid \frac{f(\zeta)}{p^s}, ..., \frac{1}{(k-1)!} f^{(k-1)}(\zeta) p^{k-1}$ we get:

$$p^s \left(\frac{f(\zeta)}{p^s} + \frac{f'(\zeta)}{p^{s-1}} \cdot \varepsilon + ... + \frac{f^{(m)}(\zeta)}{m! \cdot p^{s-m}} \cdot \varepsilon^m \right) \mod p^k$$

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- We can then take the parenthetical part and write this as a function in terms of ε

$$g(\varepsilon) := \left(\frac{f(\zeta)}{p^s} + \frac{f'(\zeta)}{p^{s-1}} \cdot \varepsilon + ... + \frac{f^{(m)}(\zeta)}{m! \cdot p^{s-m}} \cdot \varepsilon^m \right) \mod p^{k-s}$$

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- So, while it's possible to predict the number of roots via multiple cases by evaluating $ord_p(f(\zeta)), ord_p(f'(\zeta)), \ldots$, it is easier to rely on recursion for the cases where $s \in \{2, \ldots, k-1\}$.

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- Else if $s = 1$ then if $f'(\zeta) \bmod p \neq 0$ then let $count = count + 1$ (ζ has 1 unique lift);
- Else let $newf = \frac{f(\zeta)}{p^s}$; for i from 1 to q :

$$newf = newf + \frac{f^i(\zeta)}{i! \cdot p^{s-i}} \cdot x^i$$

- Let $count = count + p^{s-1} \cdot countk(newf, p, k - s)$

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- $f(\zeta + p \cdot \varepsilon) = f(-1 + 7\varepsilon) = 49\varepsilon^2 - 42\varepsilon + 231 \mod 7^4$
- Here, $s = 1$; we increment *count* by 1 and continue counting in $\mathbb{Z}/(7^3)$

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- For a polynomial with degree 84 in $\mathbb{Z}/(211^3)$:
  - Brute Force: 92.19 seconds
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- For a polynomial with degree 99 in $\mathbb{Z}/(1049^3)$:
  - Brute Force: 3.81 hours
  - Randomized Algorithm: 1000.00us

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- For a polynomial with degree 93 in $\mathbb{Z}/(104729^3)$, the randomized algorithm takes 29.00 ms.
- For a polynomial with degree 87 in $\mathbb{Z}/(179424673^3)$, the randomized algorithm takes 92.51 sec.

Analysis

Questions we want to know the answers to:

- What time complexity does this algorithm have?
- Can we can bound the maximum number of roots for any given polynomial?

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