Asymptotic Distribution of the Partition Crank

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Partition

Definition

A partition λ of $n \in \mathbb{Z}^+$ is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ such that $\lambda_1 + ... + \lambda_k = n$.
λ_i is called a part of the partition λ and $S_n = \{\lambda \text{ a partition of } n\}$.

Definition

The partition function, $p(n)$, is the distinct number of ways to write n as a sum of natural numbers in a nonincreasing order.

Example

The partitions of 4 are:

- 4
- 3+1
- 2+2
- 2+1+1
- 1+1+1+1

Thus, $p(4) = 5$.

Ramanujan Congruences

Using the function p(n), Ramanujan made the following statement:

Theorem

For any $k \in \mathbb{Z}$, we have:

$$p(5k+4) \equiv 0 \mod 5$$
$$p(7k+5) \equiv 0 \mod 7$$
$$p(11k+6) \equiv 0 \mod 11$$

as well as several other congruences modulo any number of the form $5^a 7^b 11^c$.

Rank

Freeman Dyson defined the following in order to provide a proof for Ramanujan's congruences:
For a partition λ, let

- $l(\lambda)$ = the largest part of λ

Definition

The rank of $\lambda = l(\lambda)$ - (number of parts of λ), and is denoted rank(λ).

Example (n=4)

Partitions	Rank (mod 5)	Crank (mod 5)
4	3	4
3+1	1	0
2+2	0	2
2+1+1	$-1 \equiv 4$	-2
1+1+1+1	$-3 \equiv 2$	-4

Crank

For a partition λ, let

- $\mathrm{l}(\lambda)$ = the largest part of λ
- $\mathrm{o}(\lambda)$ = the number of 1's in λ
- $\mu(\lambda)$ = the number of parts of λ larger than $\mathrm{o}(\lambda)$

Definition

The crank of $\lambda = \begin{cases} l(\lambda) & \textit{if } o(\lambda) = 0 \\ \mu(\lambda) - o(\lambda) & \textit{if } o(\lambda) > 0 \end{cases}$, and is denoted $\mathrm{crank}(\lambda)$.

Combinatorial Motivation

Example (n=6)

Partitions	Rank (mod 11)	Crank (mod 11)
6	6	6
5+1	3	0
4+2	2	4
4+1+1	1	$-1 \equiv 10$
3+3	1	3
3+2+1	0	1
3+1+1+1	$-1 \equiv 10$	$-3 \equiv 8$
2+2+2	$-1 \equiv 10$	2
2+2+1+1	$-2 \equiv 9$	$-2 \equiv 9$
2+1+1+1+1	$-3 \equiv 8$	$-4 \equiv 7$
1+1+1+1+1+1	$-5 \equiv 6$	$-6 \equiv 5$

Motivation and Goals

Definition

M(r,Q;n) $= \{\lambda \in P_n | crank(\lambda) \equiv r (mod Q)\}$

We want to:

- Prove the following statement with effective bounds on the error term: $\frac{M(r,Q;n)}{p(n)} = \frac{1}{Q} + E'(r, Q; n)$
- Prove that $\frac{M(r,Q;n)}{p(n)} \to \frac{1}{Q}$ as $n \to \infty$
- Prove surjectivity of $M(r, Q; n)$
- Prove strict log-subadditivity for the crank function

Equidistribution Theorem

Theorem

Let r and Q be relatively prime odd integers. Then

$$\frac{M(r,Q;n)}{p(n)} = \frac{1}{Q} + E'(r,Q;n),$$

where

$$\left|E'(r,Q;n)\right| \leq \left(629120 + 4.5523Q + \frac{444868}{\left(1-e^{\frac{-\pi}{Q}}\right)} + \frac{488798}{\left(1-e^{\frac{-2\pi}{Q}}\right)}\right)$$

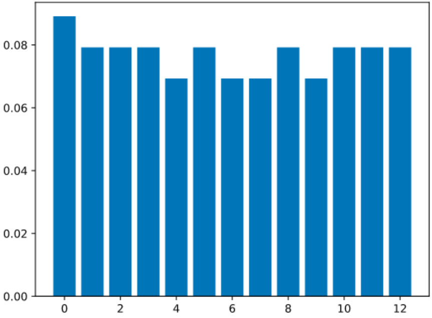
$$\times n^{\frac{7}{4}} e^{(\sqrt{24\delta_0}-1)\frac{\pi\sqrt{24n-1}}{6}}.$$

Equidistribution Corollary

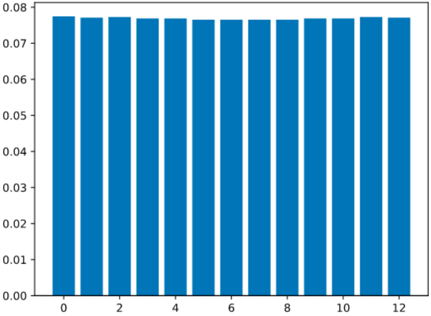
Corollary

Let r and Q be integers with Q odd. Then

$$\frac{M(r,Q;n)}{p(n)} \to \frac{1}{Q} \text{ as } n \to \infty$$



$$y = \frac{M(x,13;13)}{p(13)}$$



$$y = \frac{M(x,13;30)}{p(30)}$$

Asymptotic Equidistribution Modulo Q

Theorem

Let r and Q be relatively prime odd integers. Then

$$\left|\frac{MT_1(r,Q;n)}{p(n)}\right| \le 20.7926 n^{\frac{7}{4}} e^{\left(\sqrt{24\delta_0}-1\right)\frac{\pi\sqrt{24n-1}}{6}},$$

$$\left|\frac{MT_2(r,Q;n)}{p(n)}\right| \le (81.9414 + 4.5523Q)\, n^{\frac{7}{4}} e^{\left(\sqrt{24\delta_0}-1\right)\frac{\pi\sqrt{24n-1}}{6}},$$

and

$$\left|\frac{E(r,Q;n)}{p(n)}\right| \le \left(629016.9194 + \frac{444867.657}{\left(1-e^{\frac{-\pi}{Q}}\right)} + \frac{488797.7625}{\left(1-e^{\frac{-2\pi}{Q}}\right)}\right)$$

$$\times n^{\frac{7}{4}} e^{\left(\sqrt{24\delta_0}-1\right)\frac{\pi\sqrt{24n-1}}{6}}.$$

Surjectivity

The crank is a map such that $S_n \to \mathbb{Z} \to \mathbb{Z}/Q\mathbb{Z}$.
The map $S_n \to \mathbb{Z}/Q\mathbb{Z}$ is surjective if and only if $M(r, Q; n) > 0$ for all r:

$$M(r, Q; n) = \frac{p(n)}{Q} + MT_1(r, Q; n) + MT_2(r, Q; n) + E(r, Q; n) > 0.$$

In other words, we want to show that

$$\left|\frac{MT_1(r, Q; n)}{p(n)}\right| + \left|\frac{MT_2(r, Q; n)}{p(n)}\right| + \left|\frac{E(r, Q; n)}{p(n)}\right| < \frac{1}{Q}$$

Theorem

If $n \geq \frac{Q+1}{2}$, then given any congruence class $r \pmod{Q}$ we have

$$M(r, Q; n) > 0.$$

Strict log-Subadditivity for Crank Functions

Theorem (Ono-Bessenrodt)

If $a, b \geq 1$ *and* $a + b \geq 9$*, then*

$$p(a+b) < p(a)p(b).$$

Conjecture

For the crank function,

$$M(r, Q; a+b) < M(r, Q; a)M(r, Q; b),$$

as $a, b \to \infty$.

Generating functions

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$$= \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$

Generating functions

$$R(w,q) := \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-wq^k)(1-w^{-1}q^k)}$$

$$C(w,q) := \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) w^m q^n$$

$$=^* \prod_{n=1}^{\infty} \frac{1-q^n}{(1-wq^n)(1-w^{-1}q^n)}$$

Modular Forms

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$z, z' \in \mathbb{C}$ are $SL(2, \mathbb{Z})$ equivalent if there are integers a, b, c, d such that $ad - bc = 1$ and $z' = \frac{az+b}{cz+d}$

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Pseudo-Definition

A Jacobi Elliptic form $\vartheta(\tau, z)$ is a function which is a modular form in z for fixed τ.

Modular Forms

$$\eta(z) := q^{\frac{1}{24}}(1-q)(1-q^2)(1-q^3)\cdots$$

$$\vartheta(\tau, z) := -2\sin(\pi\tau)q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q)(1-xq)(1-x^{-1}q)$$

Modular Forms

$$P(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \frac{q^{\frac{1}{24}}}{\eta(z)}$$

$$C(w,q) = \prod_{n=1}^{\infty} \frac{1-q^n}{(1-wq^n)(1-w^{-1}q^n)} = \frac{-2\sin(\pi\tau)q^{\frac{1}{24}}\eta^2(z)}{\vartheta(\tau,z)}$$

Complex Analysis

- Unit circle
- Roots of unity $e^{2\pi i \frac{j}{k}}$
  - Primitive roots of unity
- Complex path integral
- Integrals on closed paths are 0

Cauchy's formula

Theorem

Let $f(q) = a_0 + a_1 q + a_2 q^2 + \cdots$ *be convergent inside the unit circle. Then*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(q)}{q^{n+1}} dq$$

where C *is a closed loop, has no self crossings, is contained inside the unit circle, and surrounds 0.*

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Proof.

$$\int_C \frac{f(q)}{q^{n+1}} dq$$

$$= \int_C \frac{a_0}{q^{n+1}} dq + \cdots + \int_C \frac{a_n}{q} dq + \int_C a_{n+1} dq + \cdots$$

$$= 0 + \cdots + 2\pi i a_n + 0 + \cdots$$

□

An Expression for $p(n)$

So, $p(n) = \frac{1}{2\pi i} \int_C \frac{P(q)}{q^{n+1}} dq$

Problem solved.

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