Relations Between Open and Closed Embedding Dimensions of Neural Codes

Patrick Chan, Kate Johnston

June 2020

Abstract

Neural codes are a collection of binary strings that represent the possible combinations of neurons firing within a set of receptive fields. Specifically in the case of place cells, these receptive fields are correlated to convex areas in space. Thus, when dealing with neural codes in relation to place cells, there is a desire for codes to be able to be represented with convex receptive fields. From this desire comes the need to understand the embedding dimensions in which codes require for convexity. And, since the classification of open convex neural codes are more commonly researched, this paper's purpose is to expand upon the pre-existing knowledge of already classified embedding dimensions to explore the different relationships open and closed embedding dimension. From which, we will provide the equivalent relation between the open and closed non-degenerate embedding dimensions, how non-degenerate embedding dimensions act as an upper bound for the general embedding dimensions, and lastly a hypothesis and example of the open and closed degenerate embedding dimension relation.

1 Introduction

Neural codes play a huge role in representing neural activity in the brain. In which, the study of convex neural codes have gained increased interest due to neuroscience's fascination with place cell, neurons that orient a organism's position in relation to their physical space which are represented through the composition of convex areas. In which, these convex areas will be referred to as receptive fields. And, with neural codes' relation to place cells, there is immense importance in understanding what types of spaces can be represented, with most of the focus currently being spaces compressed of all closed or all open convex receptive fields. Thus, it is necessary to understand what dimensions are needed in order for a neural code to be able to have these open or closed convex realization, where the minimally required dimensions are called embedding dimensions. In this paper, we will not only explore the relation between open and closed embedding dimension in the tradition sense of the embedding dimension, which we will refer to as the degenerate embedding dimension, but also the more novel concept of the non-degenerate embedding dimension, see Defn 3.1 & Defn 3.2. Specifically, we will given results on how the open and closed non-degenerate embedding dimensions are equal to one another, and how the degenerate embedding dimensions are less than or equal to their non-degenerate embedding dimension. We will also provide a hypothesis and a few examples on how we believe the open and closed degenerate embedding dimensions relate to one another.

2 Background

A neural code $\mathcal{C}$ of n number of receptive fields U_i is a set of code words c_j, where c_j is a binary string of length n where the digits of c_j represent the activation of neurons within the spaces of their respective receptive fields. The set of U_i is represented by $\mathcal{U}$, where $(\mathcal{U}, \mathbb{R}^d)$ denotes a realization using $\mathcal{U}$ in dimension d. Given a $(\mathcal{U}, \mathbb{R}^d)$, where $\mathcal{C} = code(\mathcal{U}, \mathbb{R}^d)$, we say that $(\mathcal{U}, \mathbb{R}^d)$ is a realization of $\mathcal{C}$. For a realization that represents a neural code $\mathcal{C}$, the specific regions laid out in $\mathcal{U}$ which represent a unique code word are called atoms $A^{\mathcal{U}}_{c_j}$.

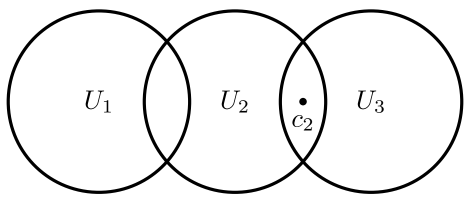

Figure 1: Realization of a code $\mathcal{C}$ in two dimensions
$\mathcal{C} = \{110, 011, 100, 010, 001, 000\}$

Example 1. *Consider the code* $\mathcal{C} = \{110, 011, 100, 010, 001, 000\}$ *with the realization of* $(\mathcal{U}, \mathbb{R}^2)$*, with* $\mathcal{U} = \{U_1, U_2, U_3\}$*. The node labeled* c_2 *represents the code word* 110*, since it is in the atom* $A^{\mathcal{U}}_{c_2} = \overline{U_1} \cap U_2 \cap U_3 = U_{\bar{1}23}$ *where* U_2 *and* U_3 *are present (indicated by a "1") and* U_1 *is absent (indicated by a "0"). So, the set of atoms being* $\{U_{12\bar{3}}, U_{\bar{1}23}, U_{1\overline{23}}, U_{\bar{1}2\bar{3}}, U_{\overline{12}3}, U_{\overline{123}}\}$*; notice the direct correlation between the set of* $A^{\mathcal{U}}_{c_j}$ *and* $\mathcal{C}$*.*

Now, having gone over the fundamentals of neural codes, we can delve into the use of them. Specifically, in the study of place cells in neuroscience, the neural code and its respective receptive fields represent convex physical spaces.

Definition 2.1. *A **convex neural code** a neural code that has a realization that can be constructed using only convex receptive fields.*

Each of these convex codes, have an embedding dimension, which is the lowest dimension such that the code can be constructed in a convex manner. To clarify, the embedding dimension of a neural code is distinct from d which represents the dimension of a singular given realization. Where as the embedding dimension of a neural code is the minimum dimension of all convex realizations of the code. When using the term embedding dimension, we refer to the neural code rather than of any particular realization.

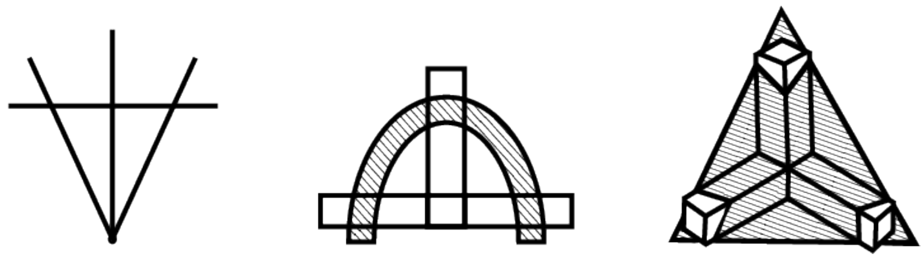

Figure 2: Closed $(\mathcal{U}, \mathbb{R}^2)$, Open $(\mathcal{U}, \mathbb{R}^2)$, Open & Closed $(\mathcal{U}, \mathbb{R}^3)$
for $\mathcal{C}_\theta = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$

Example 2. *The code* $\mathcal{C}_\theta = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$ *requires at least two dimensions to be realized. Even so,* $\mathcal{C}_\theta$ *had an open embedding dimension of three (given the shaded receptive field is forced to be non-convex in the open realization* $(\mathcal{U}, \mathbb{R}^2)$*), despite having a closed embedding dimension of two. This demonstrates how the embedding dimensions of neural codes can differ between open and closed realizations.*

So, it can be seen that the dimension needed to represent a code can change based off whether $\mathcal{U}$ is open or closed. So throughout this paper, we will classify the different types of open and closed embedding dimensions and show the relationships between them.

3 Relation Between Open and Closed Non-degenerate Embedding Dimensions

In this section we will explain our results that show the equivalent relation between the open and closed non-degenerate embedding dimensions of neural codes.

Definition 3.1. *A realization* $(\mathcal{U} = \{U_i\}, \mathbb{R}^d)$ *is* ***non-degenerate*** *if:*

1. *For any arbitrary open set* $S_o \subseteq \mathbb{R}^d$ *where* $S_o \neq \emptyset$ *and all* $A^{\mathcal{U}}_{c_j}$ *where* $A^{\mathcal{U}}_{c_j} \cap S_o \neq \emptyset$*, it is also the case that* $int(A^{\mathcal{U}}_{c_j} \cap S_o) \neq \emptyset$
2. *For all non-empty* $\sigma \subseteq [n] = \{1, 2, \cdots, n\}$, $\left(\bigcap_{i \in \sigma} \partial U_i\right) \subseteq \partial \left(\bigcap_{i \in \sigma} U_i\right)$

Definition 3.2. *The* ***open/closed non-degenerate embedding dimension*** *of a neural code, notated as* $\dim_{O\ nd}(\mathcal{C})$ *or* $\dim_{C\ nd}(\mathcal{C})$ *respectively, is the lowest dimension such that a non-degenerate realization of* $\mathcal{C}$ *can be made with open/closed convex receptive fields.*

While the relationship between the open and closed non-degenerate embedding dimensions of neural codes have not yet been explored, a lot of the groundwork has been laid by Cruz, Giusti, Itskov, and Kronhoklm in the lemma below.

Lemma 3.3. *If $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover,then:*

$$U_i \text{ are open } \implies code(\mathcal{U}, \mathbb{R}^d) = code(cl(\mathcal{U}), \mathbb{R}^d);$$
$$U_i \text{ are closed } \implies code(\mathcal{U}, \mathbb{R}^d) = code(int(\mathcal{U}), \mathbb{R}^d).$$

This states all codes that have a non-degenerate convex realization are both open and closed convex.[1]

Theorem 3.4. *Given a neural code $\mathcal{C}$, the non-degenerate closed embedding dimension $\dim_{C_{nd}}(\mathcal{C})$ and the non-degenerate open embedding dimension $\dim_{O_{nd}}(\mathcal{C})$ are equal to the same dimension d.*

Proof. Using the results from Lemma 3.3, we will provide by induction that $\dim_{C_{nd}}(\mathcal{C}) = \dim_{O_{nd}}(\mathcal{C}) = d$. Assume $\dim_{C_{nd}}(\mathcal{C}) = d_{C_{nd}}$ and $\dim_{O_{nd}}(\mathcal{C}) = d_{O_{nd}}$. There must exist some $(\mathcal{U}, \mathbb{R}^{d_{O_{nd}}})$ of the code $\mathcal{C}$. Using Lemma 3.3, we get that $\mathcal{C} = code(cl(\mathcal{U}), \mathbb{R}^{d_{O_{nd}}})$, which says there exist a closed realization of $\mathcal{C}$ in $d_{O_{nd}}$ dimensions. So, $d_{O_{nd}} \geq d_{C_{nd}}$. There must also exist some $(\mathcal{U}, \mathbb{R}^{d_{C_{nd}}})$ of the code $\mathcal{C}$. Using Lemma 3.3 once more, we get an open realization of $\mathcal{C}$ in $d_{C_{nd}}$ dimensions; $\mathcal{C} = code(int(\mathcal{U}), \mathbb{R}^{d_{C_{nd}}})$. So, it must also be the case that $d_{O_{nd}} \leq d_{C_{nd}}$. Thus, we can conclude that $\dim_{C_{nd}}(\mathcal{C}) = \dim_{O_{nd}}(\mathcal{C}) = d$. □

So, we can refer to the open and closed non-degenerate embedding dimension more generally a just the non-degenerate embedding dimension, $\dim_{nd}(\mathcal{C}) = d_{nd}$.

4 Relation Between Non-Degenerate and Degenerate Embedding Dimensions

Having established codes have a singular non-degenerative embedding dimensions regardless of open/closed-ness, see section above, we can delve into their relation to the traditional sense of the minimal embedding dimension which include open and closed degenerate realizations of neural codes. In which this section will show how the embedding dimension of a neural code that include degenerate realizations are less than or equal to its non-degenerate embedding.

Definition 4.1. *The **open/closed embedding dimension** of a neural code, notated as $\dim_{O\ d}(\mathcal{C})$ and $\dim_{C\ d}(\mathcal{C})$ respectively, is the lowest dimension such that a realization of $\mathcal{C}$ can be made with open/closed convex receptive fields regardless of degeneracy.*

Theorem 4.2. *Given a neural code $\mathcal{C}$, both the open embedding dimensions, $\dim_{O_d}(\mathcal{C})$, and the closed embedding dimension, $\dim_{Cd}(\mathcal{C})$, are less than or equal to the non-degenerate embedding dimension $\dim_{nd}(\mathcal{C})$.*

Proof. Consider the set of all convex realizations of a neural code $\mathcal{C}$, $\{(\mathcal{U}_d, \mathbb{R}^d)_h\}$. By Definition, the set of convex non-degenerate realization $\{(\mathcal{U}_{nd}, \mathbb{R}^d)_g\}$ is a subset of the convex realizations $\{(\mathcal{U}_d, \mathbb{R}^d)_h\}$. Thus, $\{(\mathcal{U}_d, \mathbb{R}^d)_h\}$ can be partitioned into two disjoint sets:

$$\{(\mathcal{U}_d, \mathbb{R}^d)_h\} = \{(\mathcal{U}_{nd}, \mathbb{R}^d)_g\} \cup \left(\{(\mathcal{U}_d, \mathbb{R}^d)_h\} \cap \overline{\{(\mathcal{U}_{nd}, \mathbb{R}^d)_g\}} \right) \tag{1}$$

So,

$$\dim_d(\mathcal{C}) = \min \left(\dim_{nd}(\mathcal{C}), \min \left(\dim \left(\{(\mathcal{U}_d, \mathbb{R}^d)_h\} \cap \overline{\{(\mathcal{U}_{nd}, \mathbb{R}^d)_g\}} \right) \right) \right) \tag{2}$$

Thus, it can be concluded that $\dim_{O_d}(\mathcal{C}) \leq \dim_{nd}(\mathcal{C})$ and $\dim_{C_d}(\mathcal{C}) \leq \dim_{nd}(\mathcal{C})$. □

This shows that the non-degenerate embedding dimension can act at the upper bound for all subsequent embedding dimensions which can be useful in providing a broad context in understanding what types of spaces a neural code can represent in the context of place cells.

5 Relation Between Open and Closed Degenerate Embedding Dimensions

The last relation that needs to be explored is the one between open and closed embedding dimensions. In which, we will show that degenerate open convex realizations are non-closed and visa versa. We will also provide a conjecture on how realizations work in dimensions greater than their embedding dimensions, and the results that can be concluded if our conjecture is proven.

But first, we would like to state the outlier cases of degenerate embedding dimensions which are non-open and non-closed neural codes. This is when there does not exist an open and/or closed convex realization of a realization, so $\dim_{O_d}(\mathcal{C}) = \infty$ and/or $\dim_{C_d}(\mathcal{C}) = \infty$ respectively. If a code is non-open closed then it will always be the case that the closed embedding dimension is less than the open embedding dimension and visa versa, with $\dim_{nd}(\mathcal{C}) = \infty$. Now, having explained the known outlier case, we can delve into our results and our conjecture along with the conjecture's possible implications if it is proven.

First, we will show our results on convex degenerate cover, which states that open degenerate realizations are not valid closed realizations of the same code, and visa versa. But, we will require the following result from [1].

Lemma 5.1. *Assume that $\mathcal{U} = \{U_i\}$ is a finite cover by convex sets. Then:*

(i) if all U_i are open and $\mathcal{U}$ satisfies Definition 3.1(2), then it also satisfies Definition 3.1(1).

(ii) if all U_i are closed and $\mathcal{U}$ satisfies Definition 3.1(1), then it also satisfies Definition 3.1(2).

Theorem 5.2. *If $\mathcal{U} = \{U_i\}$ is a convex and degenerate cover, then:*

(i) $(\mathcal{U}, \mathbb{R}^d)$ is an open realization of a neural code $\mathcal{C} \implies (cl(\mathcal{U}), \mathbb{R}^d)$ is not a closed realization of $\mathcal{C}$

(ii) $(\mathcal{U}, \mathbb{R}^d)$ is an closed realization of a neural code $\mathcal{C} \implies (int(\mathcal{U}), \mathbb{R}^d)$ is not a open realization of $\mathcal{C}$.

Proof. We prove (i) and (ii) by induction:
(i): Assume $\mathcal{U}$ is open convex and degenerate. By Lemma 5.1, there exists some $\sigma \subseteq [n] = \{1, 2, \cdots, n\}$ such that $\left(\bigcap_{i\in\sigma} \partial U_i\right) \not\subseteq \partial\left(\bigcap_{i\in\sigma} U_i\right)$. This can be rewritten as $\left(\bigcap_{i\in\sigma}(cl(U_i) \setminus U_i)\right) \not\subseteq \left(cl\left(\bigcap_{i\in\sigma} U_i\right) \setminus \left(\bigcap_{i\in\sigma} U_i\right)\right)$. So, there exists a non-empty set of elements $\{x\} \in \left(\bigcap_{i\in\sigma}(cl(U_i) \setminus U_i)\right)$ such that $\{x\} \notin \left(cl\left(\bigcap_{i\in\sigma} U_i\right) \setminus \left(\bigcap_{i\in\sigma} U_i\right)\right)$. Let $\{x\}$ be the atom for the code word $c_x = \sigma$, i.e. if $\sigma = \{1, 3\}$ then $c_x = 101$. $\left(\bigcap_{i\in\sigma}(cl(U_i) \setminus U_i)\right) = \bigcap_{i\in\sigma} cl(U_i) \cap \bigcap_{i\in\sigma} \overline{U_i}$, and $\left(cl\left(\bigcap_{i\in\sigma} U_i\right) \setminus \left(\bigcap_{i\in\sigma} U_i\right)\right) = cl\left(\bigcap_{i\in\sigma} U_i\right) \cap \left(\bigcup_{i\in\sigma} \overline{U_i}\right)$. $\bigcap_{i\in\sigma} \overline{U_i} \subseteq \bigcup_{i\in\sigma} \overline{U_i}$, so it must be the case that $A_{c_x} \in \bigcap_{i\in\sigma} cl(U_i)$, $A_{c_x} \notin cl\left(\bigcap_{i\in\sigma} U_i\right)$. Since $\bigcap_{i\in\sigma} U_i \subseteq cl\left(\bigcap_{i\in\sigma} U_i\right)$, $A_{c_x} \notin \bigcap_{i\in\sigma} U_i$. Thus, $code(\mathcal{U}, \mathbb{R}^d) \neq code(cl(\mathcal{U}), \mathbb{R}^d)$.

(ii): Assume $\mathcal{U}$ is closed convex and degenerate. By Lemma 5.1, there exists some open set $S_o \subseteq \mathbb{R}^d$ where $S_o \neq \emptyset$ and an $A^{\mathcal{U}}_{c_x}$ where $A^{\mathcal{U}}_{c_x} \cap S_o \neq \emptyset$ and $int(A^{\mathcal{U}}_{c_x} \cap S_o) = \emptyset$, i.e. $A^{\mathcal{U}}_{c_x}$ is not top dimensional. $A^{\mathcal{U}}_{c_x} = \bigcap U_i \setminus \bigcup U_j$, so $A^{\mathcal{U}}_{c_x} = \bigcap U_i \cap S$ where $S = \bigcap \overline{U_j}$ is open (and by extension top dimensional). So, it must be the case that $\bigcap U_i$ is not top dimensional. Since all U_i are closed convex, $\bigcap U_i$ must also be closed convex. So, since $\bigcap U_i$ is closed convex and not top dimensional, $int(\bigcap U_i) = \emptyset$. $int(A^{\mathcal{U}}_{c_x}) \subseteq int(\bigcap U_i) = \emptyset$, thus $int(A^{\mathcal{U}}_{c_x}) = \emptyset$ meaning a code word c_x is lost. Thus, $code(\mathcal{U}, \mathbb{R}^d) \neq code(int(\mathcal{U}), \mathbb{R}^d)$. □

This theorem acts as an extension of a theorem presented by Cruz et. al in [1] which is restated in this paper as Lemma 3.3. This gives more context to the definition of what it means to be a degenerate realization. Specifically, its contrapositive provides a nice corollary on realizations that are both open and closed convex:

Corollary 5.3. *If $\mathcal{U} = \{U_i\}$ is a convex cover and $\mathcal{C} = code(cl(\mathcal{U}), \mathbb{R}^d) = code(int(\mathcal{U}), \mathbb{R}^d)$, then $(\mathcal{U}, \mathbb{R}^d)$ is a non-degenerate realization of $\mathcal{C}$.*

Conjecture 5.4. *Let $\mathcal{C}$ have an embedding dimension of d. For all convex realizations with an embedding dimension greater than their respective neural code's embedding dimension, $\left(\mathcal{U}, \mathbb{R}^{d_\theta \geq d}\right)$, is homotopy equivalent to a realization of the neural code in the code's embedding dimension where the intermediate realizations that undergone a continuous deformation are valid realizations (convex and the code remains unchanged).*

Conjecture 5.4 has been consistent with all of our observations thus far, so we believe it likely to be true. To better illustrate the ideas of this conjecture we have provided an example below:

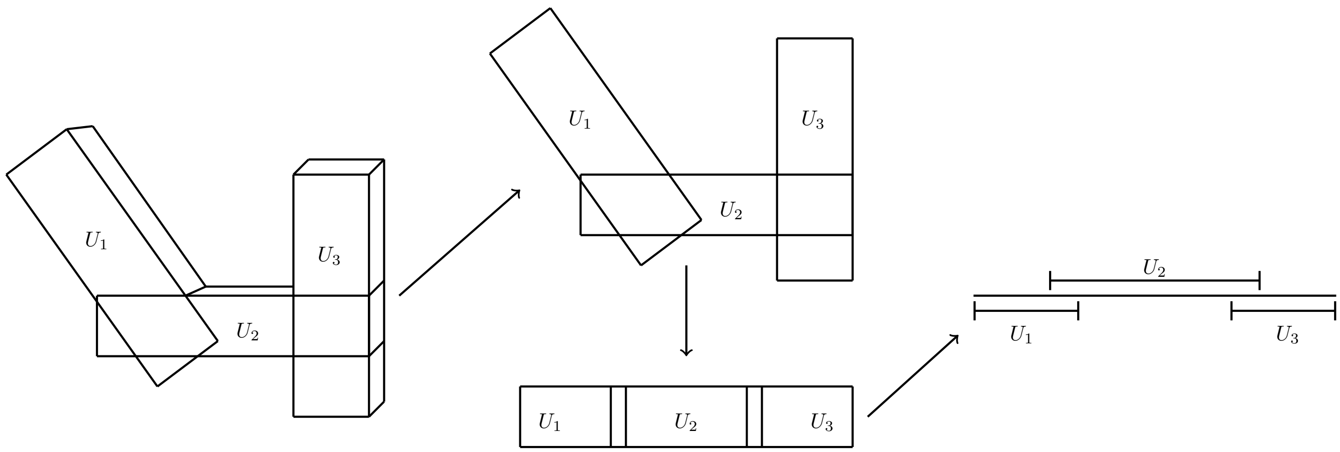

Figure 3: The continuous deformation of $\mathcal{C}$ from $d = 3$ to $d = 1$
$\mathcal{C} = \{110, 011, 100, 010, 001, 000\}$

Example 3. *Consider a realization of the neural code:*

$$\mathcal{C} = \{110, 011, 100, 010, 001, 000\}$$

in three dimensions, shown in Figure 3. We observed that $(\mathcal{U}, \mathbb{R}^3)$ *of* $\mathcal{C}$ *is homotopy equivalent to a realization in its embedding dimension of one.*

If conjecture 5.4 can be proven, it will have wide implications in understanding how open and closed degenerate embedding dimensions are related to one another.

6 Rigid Structures of Non-closed Convexity

To better clarify the outlier case of non-closed neural codes, this section we will be discussing non-closed convex neural codes, and the geometric mechanisms we have identified that make the known cases of neural codes non-closed convex; seen in [3] as C6, C10, C15.

Definition 6.1. *Let* $\mathcal{C}$ *be a neural code on* n *neurons. A subset of* $[n]$ *is* ***rigid*** *if every convex closed realization of* $\mathcal{C}l_\tau$*, a code restricted to neurons in* $\tau \subseteq [n]$*,* $\bigcup_{i \in \mathfrak{R}} U_i$ *is convex.*

Definition 6.2. *Let* $\mathcal{C}$ *be a neural code on* n *neurons. Assume that* $\mathcal{C}l_\tau$*, a code restricted to neurons in* τ*, has a* ***rigid*** *subset* $\mathfrak{R} \subseteq \tau$*. A* ***connector*** *of* $\mathfrak{R}$ *in* $\mathcal{C}$ *is a subset* $\mathfrak{C} \subseteq [n] \setminus \tau$ *such that closed convex realization of* $\mathcal{C}$*,* $\bigcap_{i \in \mathfrak{C}} U_i$ *and* *intersects two subsets atoms of $\mathcal{C}l_{\mathfrak{R}}$ such that a line segment can not be drawn between these two subsets atoms of $\mathfrak{R}$ without passing through another atom of $\mathfrak{R}$.*

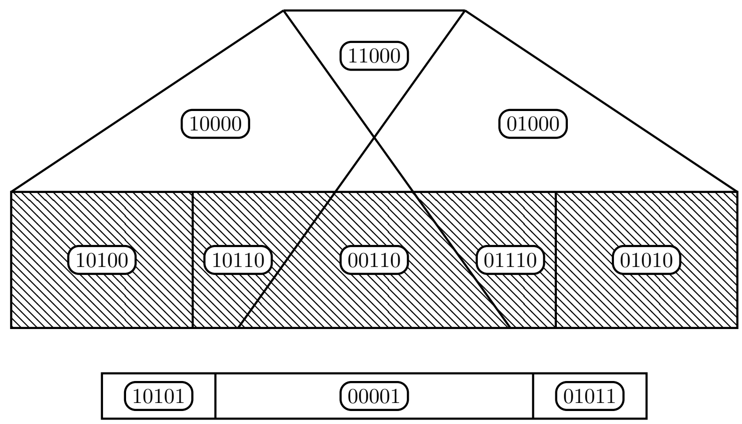

Figure 4: Top: Closed Realization of $\mathcal{C}10$ without $\mathfrak{C} = \{10101, 01011, 00001\}$ restricted to $\{U_1, U_2, U_3, U_4\}$ (rigid subset shaded); Bottom: Realization of $\mathfrak{C}$ $\mathcal{C}10 = \{10110, 10101, 01110, 01011, 11000, 10100, 01010, 00110, 10000, 01000, 00001\}$ from [3]

Example 4. *Consider the neural code:*

$$\mathcal{C}10 = \{10110, 10101, 01110, 01011, 11000, 10100, 01010, 00110, 10000, 01000, 00001\}$$

We can consider $U_3 \cup U_4$ as rigid and U_5 as the connector of the neural code.

Theorem 6.3. *If a neural code $\mathcal{C}$ contains both a rigid structure $\mathfrak{R}$ and a connector $\mathfrak{C}$ for $\mathfrak{R}$, then $\mathcal{C}$ is non-closed convex or non-convex.*

Proof. Assume for contradiction that $\{U_i\}_{i\in[n]}$ is a closed convex realization in $\mathcal{C}$. Then by definition $\bigcup_{i\in\mathfrak{R}} U_i$ is convex. Also by definition, $\bigcap_{i\in\mathfrak{C}} U_i$ intersects two atoms of $\mathfrak{R}$. Now, consider the two points $c_1, c_2 \in \mathfrak{C}$ where $\mathfrak{C}$ connects to $\mathfrak{R}$. Given our assumptions $\bigcup_{i\in\mathfrak{R}} U_i$ is convex, the line between c_1 and c_2 is completely in $\bigcup_{i\in\mathfrak{R}} U_i$. But, by definition, line segment between the two atoms are not completely contained in $\bigcap_{i\in\mathfrak{C}} U_i$. So, $\bigcap_{i\in\mathfrak{C}} U_i$ is not convex, which is a contradiction. $\square$

Theorem 6.3 can be applied to all non-closed convex neural codes seen in [3], with $C10$ shown in Figure and Example 4 above, and the rest below:

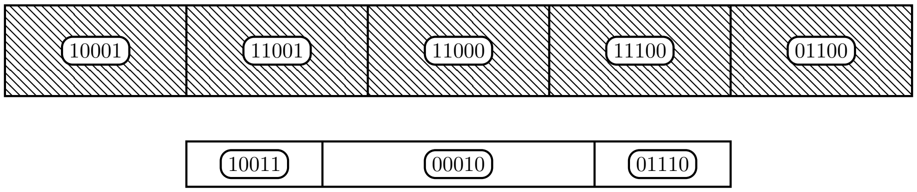

Figure 5: Top: Closed Realization of $\mathcal{C}6$ without $\mathfrak{C} = \{10011, 00010, 01110\}$ restricted to $\{U_1, U_2, U_3, U_5\}$ (rigid subset shaded); Bottom: Realization of $\mathfrak{C}$
$\mathcal{C}6 = \{11001, 11100, 10011, 01110, 10001, 11000, 01100, 00010\}$ from [3]

Example 5. *Consider the neural code:*

$$\mathcal{C}6 = \{11001, 11100, 10011, 01110, 10001, 11000, 01100, 00010\}$$

We can consider $U_1 \cup U_2 \cup U_3 \cup U_5$ as rigid and U_4 as the connector of the neural code.

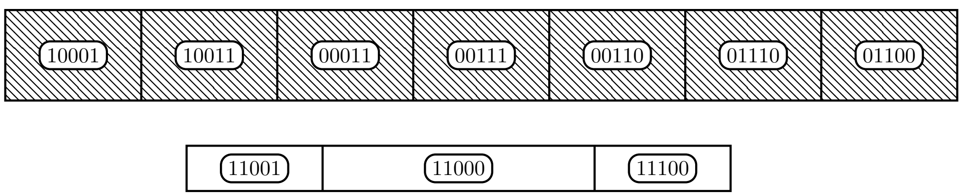

Figure 6: Top: Closed Realization of $\mathcal{C}15$ without $\mathfrak{C} = \{11001, 11000, 11100\}$ (rigid subset shaded); Bottom: Realization of $\mathfrak{C}$
$\mathcal{C}15 = \{10011, 00111, 01110, 11001, 11100, 10001, 00011, 00110, 01100, 11000\}$ from [3]

Example 6. *Consider the neural code:*

$$\mathcal{C}15 = \{10011, 00111, 01110, 11001, 11100, 10001, 00011, 00110, 01100, 11000\}$$

Unlike in Example 4 and 5, rather than a union of entire receptive fields being rigid, the rigid subset is a union of atoms. With a connector $\mathfrak{C} = \{11001, 11000, 11100\}$, $\mathfrak{R} = \mathcal{C}15 \setminus \mathfrak{C}$. Another unique aspect of C15 is that there exists multiple $(\mathfrak{R}, \mathfrak{C})$ pairs:

- $\mathfrak{C} = \{11001, 11000, 11100\}$
- $\mathfrak{C} = \{11100, 01100, 01110\}$
- $\mathfrak{C} = \{01110, 00110, 00111\}$
- $\mathfrak{C} = \{00111, 00011, 10011\}$
- $\mathfrak{C} = \{10011, 10001, 11001\}$

where in each case stated above, $\mathfrak{R} = \mathcal{C}15 \setminus \mathfrak{C}$.

7 Non-degenerate Neural Codes on up to Four Neurons

In our discussion of minimal embedding dimension we have shown that degeneracy is an important characteristic in a neural code. It is then in our interest to understand which codes are known to be non-degenerate. In this section we present results on a classification of neural codes that are non-degenerate.

Theorem 7.1. *Let $\mathcal{C}$ be a neural code with no local obstructions on $n \leq 4$ neurons. Then, $\mathcal{C}$ is realizable by a convex, non-degenerate cover.*

Proof. Curto et. al. [2] showed that for codes up on up to 4 neurons, no local obstructions is equivalent to max-intersection complete. By Theorem 1.2 and Theorem 2.12 in [1] a neural code $\mathcal{C}$ that is max intersection-complete is realizable by a convex, non-degenerate cover. Therefore all codes on $n \leq 4$ neurons are realizable by a convex, non-degenerate cover. □

8 Discussion

In this paper we identified a method of classifying embedding dimensions of neural codes based on the degeneracy of the realizations, and found the relationships between them. From which, we showed results on how there is only a singular non-degenerate embedding dimension (shown in Theorem 3.4), how the non-degenerate embedding dimension acts as an upper bound for all other embedding dimension (shown in Theorem 4.2), and provided a Conjecture 5.4 that provides an outline for how we believe realizations behave in a dimension greater than their neural code's embedding dimension. All of which, we believe will aid in the goal of understand what types of spaces can a neural code represent in relation to place cells.

If one is searching for the next question for exploration, we believe it a worthwhile task would be to understand which neural codes have non-degenerate realizations, i.e. if a code is open and closed convex, does that imply the existence of a non-degenerate realization?

Acknowledgements

Special thanks to Alex Ruys de Perez and Dr. Anne Shiu for their invaluable mentorship in this research project, for providing the direction, guidance, and insight to make this paper a reality.

This research was conducted as a part of the NSF-funded REU (DMS-1757872).

References

[1] Joshua Cruz, Chad Giusti, Vladimir Itskov, and Bill Kronholm. On open and closed convex codes. *Discrete & Computational Geometry*, 61(2):247–270, 2019.

[2] Carina Curto, Elizabeth Gross, Jack Jeffries, Katherine Morrison, Mohamed Omar, Zvi Rosen, Anne Shiu, and Nora Youngs. What makes a neural code convex? *SIAM Journal on Applied Algebra and Geometry*, 1.

[3] Phillipson Goldrup. Classification of open and closed convex codes on five neurons. *Advances in Applied Mathematics*, 112:101948, 2020.