Degenerate and Non-Degenerate Embedding Dimensions of Neural Codes

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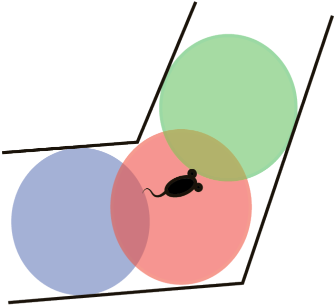
Texas A&M REU 2020

July 21st 2020

Motivation

Place Cells

Place Cells are a collection of neurons that relay information on an organism's spacial position within a location.



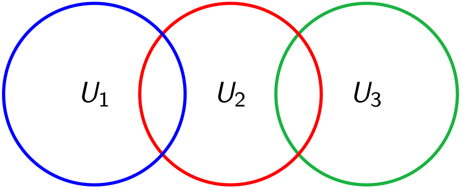
Ultimate Goal:

Understand what types of spaces a given neural code can represent.

Background:

Neural Code

A **Neural Code** is collection of code words that represent the possible combinations of neurons firing within a set of given receptive fields.



Example

Set of Receptive Fields: $\mathcal{U} = \{U_1, U_2, U_3\}$

$\mathcal{C} = \{\underline{000}, \underline{100}, \underline{010}, \underline{001}, \underline{110}, \underline{011}\}$

$A_{011}^{\mathcal{U}} = \{(U_2 \cap U_3) \setminus U_1\}$ $\mathcal{C} = code(\mathcal{U}, \mathbb{R}^2)$

Convexity

A set is **convex** if for any two points within the set, the line segment between them can be drawn wholly within the set.

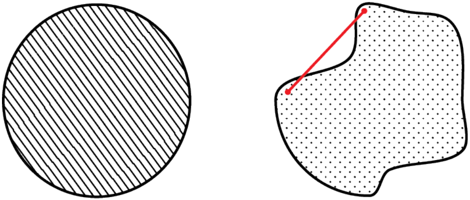


Figure: Convex (left), Non-convex (right) d=2

Closed/Open Convexity in Neural Codes

A **closed/open convex neural code** is a neural code that can be represented as a set of *convex* receptive fields where the receptive fields are closed/open sets.

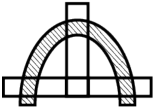
Figure: U_1:(Blue) ; U_2:(Red)

Example

$\mathcal{U} = \{U_1, U_2\}$
$\mathcal{C}_{\mathcal{C}} = \{00, 10, 01, 11\}$ $\quad \mathcal{C}_{\mathcal{C}} = code(cl(\mathcal{U}), \mathbb{R}^1)$
$\mathcal{C}_{\mathcal{O}} = \{00, 10, 01\}$ $\quad \mathcal{C}_{\mathcal{O}} = code(int(\mathcal{U}), \mathbb{R}^1)$

Open/Closed – Embedding Dimension

The **open/closed embedding dimension** is the lowest dimension such that a neural code has an open/closed-convex realization



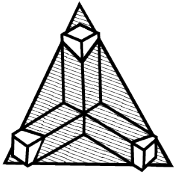


Figure: Left: Non-Convex d=2 | Right: Convex d=3

Example

$\mathcal{C}_{\mathcal{O}} = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$

Project:

Find and classify the relationships between the minimal open and closed embedding dimensions.

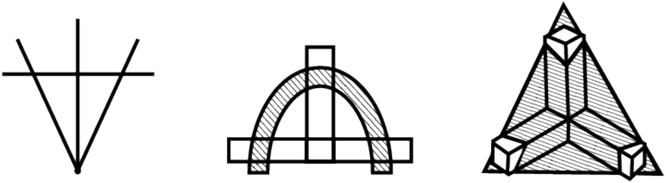


Figure: Closed $(\mathcal{U}, \mathbb{R}^2)$, Open $(\mathcal{U}, \mathbb{R}^2)$, Open & Closed $(\mathcal{U}, \mathbb{R}^3)$ for $\mathcal{C} = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$

Non-degenerate Realizations

A realization $(\mathcal{U} = \{U_i\}, \mathbb{R}^d)$ is **non-degenerate** if:

1. For any arbitrary open set $S_o \subseteq \mathbb{R}^d$ where $S_o \neq \emptyset$ and all $A^{\mathcal{U}}_{c_j}$ where $A^{\mathcal{U}}_{c_j} \cap S_o \neq \emptyset$, it is also the case that $int(A^{\mathcal{U}}_{c_j} \cap S_o) \neq \emptyset$
2. For all non-empty $\sigma \subseteq [n] = \{1, 2, \cdots, n\}$, $\left(\bigcap_{i\in\sigma} \partial U_i\right) \subseteq \partial\left(\bigcap_{i\in\sigma} U_i\right)$

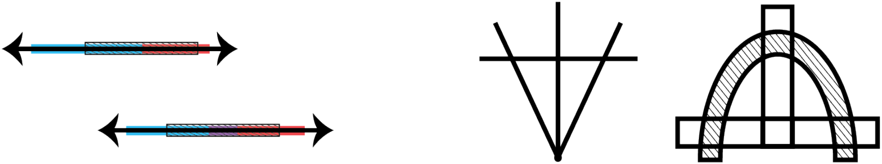
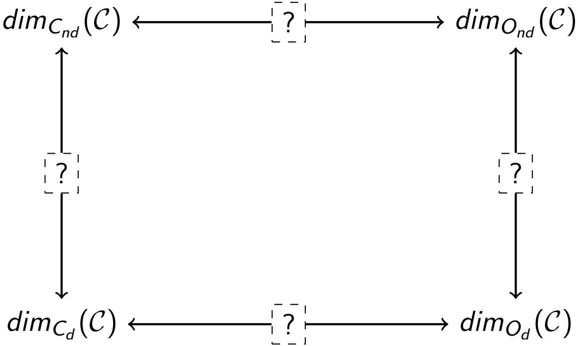


Figure: Left $(\mathcal{U}, \mathbb{R}^1)$; Right: $(\mathcal{U}, \mathbb{R}^2)$

Classification of Embedding Dimensions

- Open Non-Degenerate Embedding Dimension $dim_{O_{nd}}(\mathcal{C}) = d_{O_{nd}}$
- Closed Non-Degenerate Embedding Dimension $dim_{C_{nd}}(\mathcal{C}) = d_{C_{nd}}$
- Open Degenerate Embedding Dimension $dim_{O_d}(\mathcal{C}) = d_{O_d}$
- Closed Degenerate Embedding Dimension $dim_{C_d}(\mathcal{C}) = d_{C_d}$



Question 1:

What is the relation between the open non-degenerate and the closed non-degenerate embedding dimension?

Non-degenerate Embedding Dimension

The **non-degenerate embedding dimension** of $\mathcal{C}$ is the lowest dimension such that a convex non-degenerate realization can be made.

Lemma 1: (J. Cruz, C. Giusti, V. Itskov, and B. Kronholm)

If $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover,then:

$$U_i \text{ are open } \implies code(\mathcal{U}, \mathbb{R}^d) = code(cl(\mathcal{U}), \mathbb{R}^d);$$
$$U_i \text{ are closed } \implies code(\mathcal{U}, \mathbb{R}^d) = code(int(\mathcal{U}), \mathbb{R}^d).$$

This states all codes that have a non-degenerate convex realization are both open and closed convex.

Lemma 1: (J. Cruz, C. Giusti, V. Itskov, and B. Kronholm)

If $\mathcal{U} = \{U_i\}$ is a convex and non-degenerate cover,then:

$$U_i \text{ are open } \implies code(\mathcal{U}, \mathbb{R}^d) = code(cl(\mathcal{U}), \mathbb{R}^d);$$
$$U_i \text{ are closed } \implies code(\mathcal{U}, \mathbb{R}^d) = code(int(\mathcal{U}), \mathbb{R}^d).$$

Theorem 1: (Chan-Johnston)

Given a neural code $\mathcal{C}$, the non-degenerate closed embedding dimension $\dim_{C_{nd}}(\mathcal{C})$ and the non-degenerate open embedding dimension $\dim_{O_{nd}}(\mathcal{C})$ are equal to the same dimension d.

Question 2:

What is the relation between the non-degenerate and the degenerate embedding dimension?

Degenerate Embedding Dimension

The **degenerate embedding dimension** of $\mathcal{C}$ is the lowest dimension such that a convex realization can be made regardless of degeneracy.

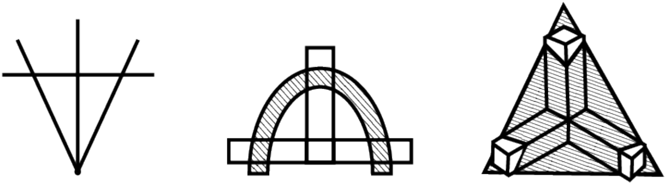


Figure: Closed $(\mathcal{U}, \mathbb{R}^2)$, Open $(\mathcal{U}, \mathbb{R}^2)$, Open & Closed $(\mathcal{U}, \mathbb{R}^3)$ for $\mathcal{C} = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$

Theorem 2: (Chan-Johnston)

Given a neural code $\mathcal{C}$, the non-degenerate embedding dimension $dim_{nd}(\mathcal{C})$ is greater than or equal to the degenerate open and closed embedding dimension, $dim_{O_d}(\mathcal{C})$ and $dim_{C_d}(\mathcal{C})$.

Importance

The non-degenerate embedding dimension acts as an upper bound for all other embedding dimensions.

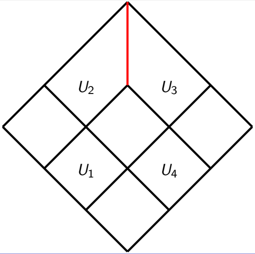
Question 3:

What is the relation between open and closed degenerate embedding dimension?

Theorem 3: (Chan-Johnston)

If $\mathcal{U}$ is a convex and degenerate cover:

1. $(\mathcal{U}, \mathbb{R}^d)$ is an open realization of a neural code $\mathcal{C} \implies (cl(\mathcal{U}), \mathbb{R}^d)$ is not a closed realization of $\mathcal{C}$
2. $(\mathcal{U}, \mathbb{R}^d)$ is an closed realization of a neural code $\mathcal{C} \implies (int(\mathcal{U}), \mathbb{R}^d)$ is not a open realization of $\mathcal{C}$.



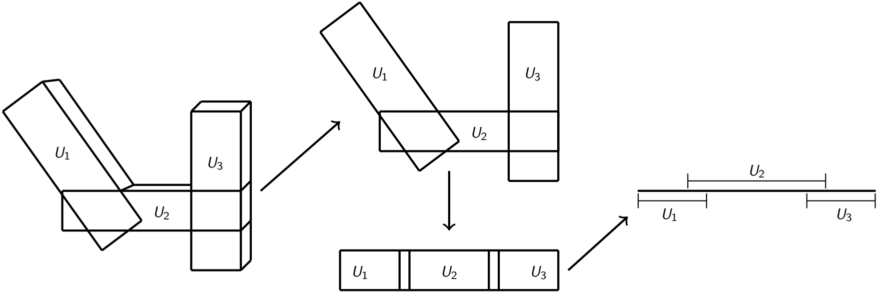


Figure: The continuous deformation of $\mathcal{C}$ from d=3 to d=1
$\mathcal{C} = \{110, 011, 100, 010, 001, 000\}$

Conjecture 1: (Chan-Johnston)

Let $\mathcal{C}$ have an embedding dimension of d. For all convex realizations with an embedding dimension greater than their respective neural code's embedding dimension, $(\mathcal{U}, \mathbb{R}^{d_\theta \geq d})$, is homotopy equivalent to a realization of the neural code in the code's embedding dimension where the intermediate realizations that have undergone a continuous deformation are valid realizations (convex and the code remains unchanged).

Theorem 4: (Chan-Johnston)

Let $\mathcal{C}$ be a neural code that satisfies Conjecture 1. Then, if a neural code $\mathcal{C}$ is open and closed convex and there exists a non-degenerate realization, then:

1. $\dim_{nd}(\mathcal{C}) = \dim_{O_d}(\mathcal{C})$,
2. $\dim_{O_d}(\mathcal{C}) \geq \dim_{C_d}(\mathcal{C})$.

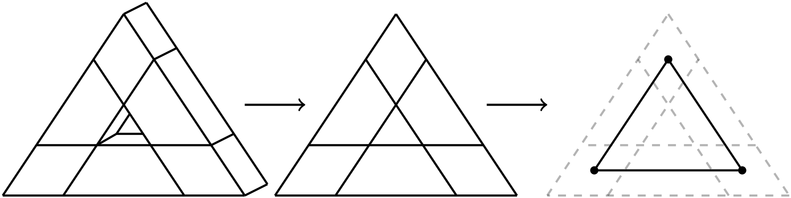


Figure: Continuous deformation of a non-degenerate realization $\mathcal{C} = \{110, 101, 011, 100, 010, 001\}$

Theorem 4: (Chan-Johnston)

Let $\mathcal{C}$ be a neural code that satisfies Conjecture 1. Then, if a neural code $\mathcal{C}$ is open and closed convex and there exists a non-degenerative realization, then:

1. $\dim_{nd}(\mathcal{C}) = \dim_{O_d}(\mathcal{C})$,
2. $\dim_{O_d}(\mathcal{C}) \geq \dim_{C_d}(\mathcal{C})$.

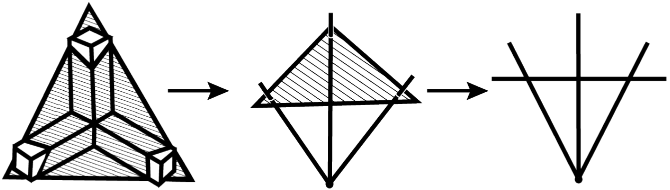


Figure: Continuous deformation of a non-degenerative realization $\mathcal{C} = \{0000, 1000, 0100, 0010, 0001, 1001, 0101, 0011, 1110\}$

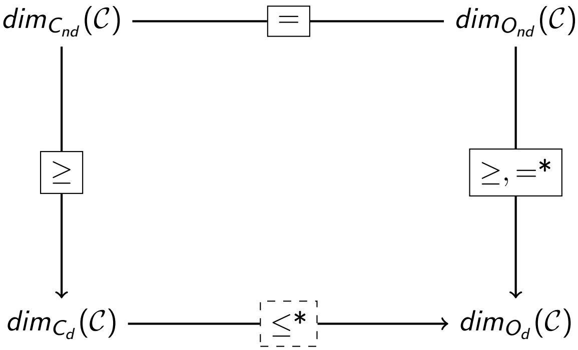


Figure: *Assumes Conjecture 1 and the existence of a non-degenerate realization

Possible Future Questions:

- Prove conjecture 1.
- If a neural code is open and closed convex, then does there exist a non-degenerate realization?

THANK YOU!

Mentors: Dr.Anne Shiu, Alex Ruys de Perez, Dr. Ola Sobieska

Funding: National Science Foundation