UNDERGRADUATE RESEARCH SCHOLAR

Geodesic Equivalence in sub-Riemannian Geometry

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Some Preliminaries: Riemannian Metrics

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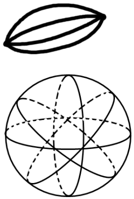

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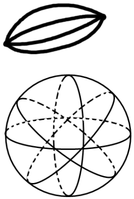

More abstractly, a Riemannian structure (M, g) on a smooth manifold is given by choosing an inner product g_p on the tangent space T_pM for any $p \in M$ smoothly on p.
$\Rightarrow$ one can define the length of a curve w.r.t g and the notion of geodesics.

Geodesic Equivalence

Two Riemannian Structures (M_1, g_1) and (M_2, g_2) are geodesically equivalent if there is a diffeomorphism $F: M_1 \to M_2$ which sends any geodesic of (M_1, g_1) to a geodesic on (M_2, g_2) as unparametrized curves

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Geodesic Equivalence cont.

A trivial way to produce a metric, which is geodesically equivalent to another given metric, is to multiply it by a constant, i.e $g_1 = \mathcal{K} g_2$ where $\mathcal{K} \in \mathbb{R}$

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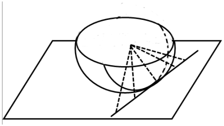

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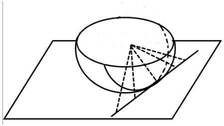

All pairs of locally geodesically equivalent Riemannian metrics with regularity assumption -Levi-Civita (1896).

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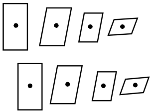

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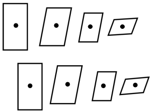

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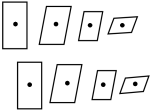

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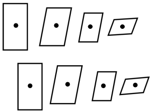

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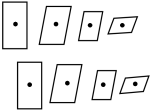

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Applications: Motion planning of car-like robots.

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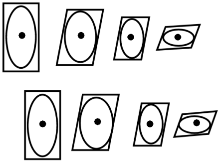

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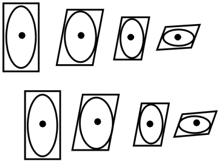

Riemannian case: $D(p) = T_P M$

Transition operator and the Cauchy characteristics

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Given two sub-Riemannian structures (M, D, g_1) and (M, D, g_1) the transition operator S_p at the point p from one structure to another is the linear operator $S_p : D(p) \mapsto D(p)$ satisfying

$$g_{2p}(v_1, v_2) = g_{1p}(S_q v_1, v_2), \quad v_1, v_2 \in D(p).$$

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A Cauchy characteristic subspace $C(p)$ is the following subspace of $D(p)$: $X \in D_p$ if for any vector field $\tilde{X}$ tangent to D and with $\tilde{X}(p) = X$ we have $[\tilde{X}, D](p) \subset D(p)$.

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A point p_0 is called regular if S_p has the same number of distinct eigenvalues in a neighborhood of p and $\dim\ C(p)$ is constant in the same neighborhood. In this case C is called the Cauchy characteristic subdistribution of D.

Conjecture

The general goal: To describe all pairs of locally geodesically equivalent sub-Riemannian metrics in a neighborhood of regular points.

Conjecture (A.C.-Zelenko)

Let (M, D, g_1) and (M, D, g_2) be sub-Riemannian structures having the same geodesics up to reparametrization and p_0 be a regular point w.r.t these metrics. Let $C^{\perp}$ be a subdistribution of D obtained by taking the orthogonal complement of C with respect to the inner product g_1. Then the following statements hold in a neighborhood of p_0

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- *The fiber $C(p)$ the Cauchy characteristic distribution C is an invariant subspace of the transition operator S_p for any p in a neighborhood of p_0.*

Conjecture (continued)

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- *The fiber $C^{\perp}(p)$ belongs to one eigenspace of the transition operator S_p*

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- *There is a natural number* ℓ *such that the distribution* $(C^{\perp})^{\ell}$*, spanned by all the iterative Lie brackets of the length not greater than* ℓ *of the vector field tangent to* $C^{\perp}$*, is involutive and* $TM = (C^{\perp})^{\ell} \oplus C$.

Conjecture(continued)

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- *Assume that $k_1, \ldots, k_m$ are the multiplicities of the eigenvalues of S_p, which are different from the eigenvalue corresponding to $C^\perp$ and let $k_0 = \dim M - \sum_{i=1}^{m} k_i$. Then there exists a local coordinate system $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$, where $\bar{x}_i = (x_i^1, \ldots, x_i^{k_i})$ such that the quadratic forms of the inner products g_1 and g_1 have the form*

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$$g_1(\dot{\bar{x}}, \dot{\bar{x}}) = \sum_{s=0}^{k} \gamma_s(\bar{x}) b_s(\dot{\bar{x}}_s, \dot{\bar{x}}_s),$$

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where the velocities $\dot{\bar{x}}$ belong to D,

$$\lambda_s(\bar{x}) = \beta_s(\bar{x}_s) \prod_{l=0}^{k} \beta_l(\bar{x}_l),$$

$$\gamma_s(\bar{x}) = \prod_{l \neq s} \left| \frac{1}{\beta_l(\bar{x}_l)} - \frac{1}{\beta_s(\bar{x}_s)} \right|,$$

$\beta_s(p_0) \neq \beta_l(p_0)$ for all $s \neq l$ and β_s is constant if $k_s \geq 1$.

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- The case of quasi-contact (even-contact) distributions, i.e. when $\mathrm{rank} D = \dim\ M - 1$, $\dim M$ is even , and $\mathrm{rank} C = 1$ (I. Zelenko, 2004)

The main result of the project

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Now we are working on the validation of the conjecture in the general case of corank 1 distributions, i.e. when $\mathrm{rank}\, D = \dim M - 1$ and without restrictions on the rank of the Cauchy characteristic sub-distribution C.

The main steps of the proof

- Using the Hamiltonian formalism of the Pontryagin Maximum Principle of Optimal Control Theory we reformulate the problem of geodesic equivalence in terms of an orbital equivalence of the corresponding sub-Riemannian Hamiltonian systems and we get an overdetermined system of equation for the orbital diffeomorphism.

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- Interpreting these divisibility conditions in geometric terms by means of the classical Frobenius theorem (on integrability of involutive distributions) and using the separation of variables technique, we got the result.

THANK YOU FOR YOUR ATTENTION

Please enjoy your day
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