Convergence Preserving Permutations and Divergent Fourier Series

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Fourier Series: Introduction

The Fourier series of a continuous function $f(\theta)$ on the interval $[-\pi, \pi]$ is

$$\widetilde{f}(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

where

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

Fejér's Example

$$F(x) = \sum_{k=1}^{\infty} \alpha_k Q_{N_k},$$

where $\alpha_k = k^{-2}$, $N_k = 2^{k^3}$, and

$$Q_N(x) = e^{2iNx} \sum_{\substack{j=-N \\ j\neq 0}}^{N} \frac{e^{ijx}}{j}.$$

Question

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Yes!

Definition (Cesáro Means)

The Cesáro means of a sequence $\{a_n\}$ are the terms of the sequence $\{c_n\}$, where

$$c_n = \frac{1}{n} \sum_{j=1}^{n} a_j.$$

In other words, the arithmetic mean of of the first n elements of $\{a_n\}$.

Another useful tool to fix these divergence issues are λ-permutations.

Definition

A permutation σ of $\mathbb{N}$ is said to be a λ-permutation

1.) if $\sum_{j=1}^{\infty} a_j$ converges, then so does $\sum_{j=1}^{\infty} a_{\sigma(j)}$;
2.) there exists a divergent series $\sum_{j=1}^{\infty} b_j$ such that $\sum_{j=1}^{\infty} b_{\sigma(j)}$ converges.

We denote the set of all such permutations as Λ.

Blocks

A block consists of consecutive integers

$$[c, d]_N = \{x \in \mathbb{Z}^+ : c \leq x \leq d\},$$

where N is the block number of the union of disjoint blocks for a sequence. For example,

$$\{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_8\} = \{2, 3, 5, 9, 1, 6, 10, 11\}$$

$$= [1, 3]_N \bigcup [5, 6]_N \bigcup [9, 11]_N$$

Then we say that the block number sequence for σ in this case is 3.

Vellman furthered the conditions in [Vel06] for σ to be considered a λ-permutation.

Theorem (Velleman, 2006)

3.) For a λ-permutation σ, the block number sequence for σ is bounded.

An Example

We have mentioned how some Fourier series may diverge, and two ways in which they may be fixed to converge. Now we present a specific example of a function F that has the following properties:

1.) $F(x)$ is continuous at $x = 0$.

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1.) $F(x)$ is continuous at $x = 0$.

2.) The partial sums of the Fourier series of $F(x)$ diverges at $x = 0$.

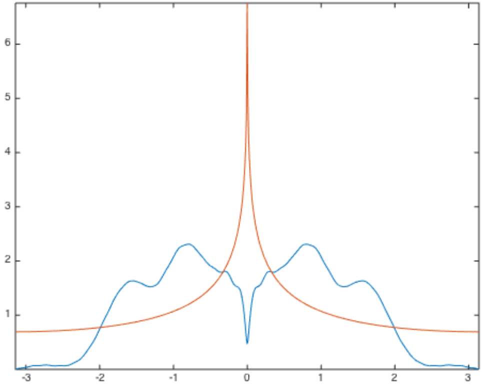
Construction:

$$F(x) = \sum_{k=1}^{\infty} \alpha_k Q_{N_k},$$

where $\alpha_k = k(log(k))^{2.1}$, $N_k = \left\lceil (1.1)^{k(log(k))^{2.1}} \right\rceil$, and

$$Q_N(x) = e^{2iNx} \sum_{\substack{j=-N \\ j \neq 0}}^{N} \frac{e^{ijx}}{j}.$$

The graph shows 5 partial sums of $|F(x)|$ and $|S_{N_k}(F,x)|$



Theorem (McNeal-Zeytuncu,2005)

There exists a λ-permutation, σ, such that

$$\lim_{n\to\infty} S_{\sigma(n)}(F, x)$$

exists for all $x \in [-\pi, \pi]$.

Main Question

Can we fix *any* divergent Fourier series with a λ-permutation?

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1.) $G(x)$ is continuous on $[-\pi, \pi]$.
2.) The partial sums of the Fourier series of $G(x)$ diverge at $x = 0$.
3.) However, there exists no λ-permutation σ that fixes its divergence issue.

Theorem

There exists a continuous function $G(x)$ such that

1.) $\limsup_{n\to\infty} S_n(G,0) = \infty$, *and*

2.) $\lim_{n\to\infty} S_{\sigma(n)}(G,x)$ *does not exist for all* $\sigma \in \Lambda$.

Series

For any $N \in \mathbb{N}$, permute the integers $\{1, 2, \cdots, 2N\}$ as

$$\{2N, 1, 2N-1, 2, 2N-2, 3, \cdots, N+2, N-1, N+1, N\}$$

we label this permutation by η.

Construction:

Let $\beta_k = \frac{1}{k^2}$ and $N_k = 2^{k^3}$ now

$$G(x) = \sum_{k=1}^{\infty} \beta_k \widetilde{Q}_{N_k}$$

For any even positive integer $\mathbb{N}$, define

$$\widetilde{Q}_N(x) = \left(\sum_{j=1}^{N} \frac{\exp(i(N+j-1)x)}{j} - \sum_{j=1}^{N} \frac{\exp(i(2N+j)x)}{\eta(j)} \right)$$

Future Work

Cesaro Means vs. λ-Permutations, in terms of computational efficiency.

References

[Kör83] T. W. Körner. The behavior of power series on their circle of convergence. In *Banach spaces, harmonic analysis, and probability theory (Storrs, Conn., 1980/1981)*, volume 995 of *Lecture Notes in Math.*, pages 56–94. Springer, Berlin, 1983.

[MZ06] Jeffery D. McNeal and Yunus E. Zeytuncu. A note on rearrangement of Fourier series. *J. Math. Anal. Appl.*, 323(2):1348–1353, 2006.

[Vel06] Daniel J. Velleman. A note on λ-permutations. *Amer. Math. Monthly*, 113(2):173–178, 2006.

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Thank you!