Linear Transport Equations with Continuous Solutions

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Abstract. Linear transport equations with continuous solutions are considered. The main results are: existence of a weak solution imposing minimal restrictions on the transport velocity and uniqueness via stability provided that the Filippov flow of the associated ODE is unique.

§1. Introduction

We are interested in the initial value problem

\begin{align*}
    \begin{cases}
        u_t + a \cdot \nabla_x u = 0, & (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^N, \\
        u(0, \mathbf{z}) = u^0(\mathbf{z}), & \mathbf{z} \in \mathbb{R}^N,
    \end{cases}
\end{align*}

where \( \mathbf{z} := (x_1, \ldots, x_N) \) and \( a(t, \mathbf{z}) := (a_1(t, \mathbf{z}), \ldots, a_N(t, \mathbf{z})) \in (L^1_{\text{loc}}([0, T] \times \mathbb{R}^N))^N \). There are two main approaches to (1). The first one is to assume enough regularity from the transport velocity \( a \) which will guarantee two sided uniqueness of the flow associated with (1). In this case, using regularization arguments for (1), it is possible to show that there exists a unique flow which is measure preserving and the solution \( u(t, \cdot) \) of (1) at time \( t \) is the image of the initial data \( u^0 \) under the flow. The main result of this type is the paper of DiPerna and Lions [7], see also [6]. The second approach is to consider the case in which the associated ODE can possibly have many flow solutions. This is also the case that comes up if we consider transport equations associated with conservation laws, see [3,4,5,12,17,18]. In general, the transport velocity \( a \) can be discontinuous with the only restriction (3). In this second approach, we need to impose some regularity on the initial data and require this regularity to be preserved by the solution of (1). In this paper we take the second approach. We study whether for a given initial condition \( u^0 \in C(\mathbb{R}^N) \), there is a solution to (1) from \( C([0, T] \times \mathbb{R}^N) \). The main difficulty we should encounter is that for certain velocities \( a \) we can have no solutions or infinitely many solutions.
with the same initial data. Therefore, we need to impose conditions on \( a \) to have existence of a solution to (1). Once existence is established, a criteria is needed to extract a unique solution.

We are going to consider the problem (1) in the weak sense. We say \( u \in C([0, T] \times \mathbb{R}^N) \) is a weak solution to (1) if

\[
\int_0^T \int_{\mathbb{R}^N} u \phi_t + u \text{div}_x(\mathbf{a} \phi) \, dz \, dt = \int_{\mathbb{R}^N} u(T, x) \phi(T, x) \, dz - \int_{\mathbb{R}^N} u_0(x) \phi(0, x) \, dz,
\]

for all test functions \( \phi \in C^\infty([0, T] \times \mathbb{R}^N) \) with compact support. Since we impose that the solution \( u \) be in \( C([0, T] \times \mathbb{R}^N) \), we can treat fairly general transport velocities \( \mathbf{a} \) in the weak formulation (2). For example, if we denote by \( D_i \) the distributional derivative with respect to \( x_i \), it is enough to have that \( D_i a_i \) is a Radon (signed) measure, \( D_i a_i \in \mathcal{M}_{loc}([0, T] \times \mathbb{R}^N) \) for \( 1 \leq i \leq N \), hence

\[
D_i a_i \text{ is a bounded measure on } [0, T] \times K
\]

for all compact sets \( K \subset \mathbb{R}^N, 1 \leq i \leq N \). Recall that a Borel measure \( \mu \) on \([0, T] \times \mathbb{R}^N\) is called a Radon measure if each subset of \([0, T] \times \mathbb{R}^N\) is contained within a Borel set of equal \( \mu \) measure and that \( \mu(P) < \infty \) for each compact set \( P \subset [0, T] \times \mathbb{R}^N \), see [19] for more details. If \( \mathbf{a} \) satisfies (3), it is easy to show that \( \text{div}_x(\mathbf{a} \phi) \in \mathcal{M}_{loc}([0, T] \times \mathbb{R}^N) \) and hence the integrals in (2) are defined.

Our goal is to develop an existence, uniqueness, and regularity theory in the multidimensional case, similar to the one in [1,8,14]. The condition (3), by itself, is not enough for existence of continuous solutions to (2), see [17], and we need to impose additional regularity on the transport velocity \( \mathbf{a} \) to have an existence-uniqueness theory. A brief outline of this paper is the following. We start with results for ODEs in Section 2 and derive a stability result for the Filippov flow (Theorem 7). In section 3, we prove existence and stability results for linear transport equations. Our main result is Theorem 11.

§2. ODEs with discontinuous coefficients

In this section, we recall the definition and the basic properties of ordinary differential equations as introduced by Filippov in [9]. Filippov’s results are formulated for a general domain. He also developed a concept of uniqueness on the right; this means that the solution is unique forward in time. We will need the analogous notion of left uniqueness when working with PDEs in the next section. We will reformulate Filippov’s results in our setting which is slightly different than Filippov’s original work but similar to other authors (see [1,14,15]).

For brevity, we use the notation \( \mathbf{z} := (x_1, \ldots, x_N) \in \mathbb{R}^N \) and \( (t, \mathbf{z}) := (t, x_1, \ldots, x_N) \in \Omega := [0, T] \times \mathbb{R}^N \). We will often call \( \mathbf{z} \) a vector and denote by \( |\mathbf{z}| \) its norm

\[
|\mathbf{z}| := (x_1^2 + \ldots + x_N^2)^{1/2},
\]
and by \( \mathbf{z} \cdot \mathbf{y} \) the scalar product of the vectors \( \mathbf{z} \) and \( \mathbf{y} \). We will use the usual Euclidean distance in \( \mathbb{R}^N \) and \( \Omega \), that is \( |\mathbf{z} - \mathbf{y}| \) and

\[
| (t', \mathbf{z}) - (t'', \mathbf{y}) | := ((t' - t'')^2 + |\mathbf{z} - \mathbf{y}|^2)^{1/2}.
\]

We also use the vector notation \( \mathbf{z} \leq \mathbf{y} \) if \( x_i \leq y_i \) for all \( 1 \leq i \leq N \) (similarly for all other inequalities). We will use bold face for vectors and vector functions, subscripts for their coordinates, and superscripts for all sequences.

Let \( \mathbf{a} := \mathbf{a}(t, \mathbf{z}) \) be a given vector field on \( \mathbb{R}^N \), i.e. \( \mathbf{a} \) is defined on \([0, T] \times \mathbb{R}^N \) and takes values in \( \mathbb{R}^N \). We assume that \( \mathbf{a} \in (L^1_{loc}([0, T] \times \mathbb{R}^N))^N \) and that there exists a function \( m \in L^1[0, T] \) such that

\[
|\mathbf{a}(t, \mathbf{z})| \leq m(t), \quad \text{a.e. } (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^N.
\]

We consider the system of ordinary differential equations in vector form

\[
\frac{d\mathcal{X}}{ds} = \mathbf{a}(s, \mathcal{X}), \quad 0 \leq s \leq T,
\]

where \( \mathcal{X}(\cdot; (t, \mathbf{z})) \) is a vector function from \([0, T]\) to \( \mathbb{R}^N \). Let’s recall the definition of generalized characteristics introduced by Filippov in [9].

**Definition 1.** A generalized solution to (5) in the sense of Filippov or in short a Filippov characteristic \( \mathcal{X}(s; (t, \mathbf{z})) \), passing through \( \mathbf{z} \in \mathbb{R}^N \) at time \( t \in [0, T] \), is an absolutely continuous function \( \mathcal{X} \) \( \cdot := \mathcal{X}(\cdot; (t, \mathbf{z})) \) which satisfies

(i) \( \mathcal{X}(t; (t, \mathbf{z})) = \mathbf{z} \),

(ii) For almost all \( s \in (0, T) \) and for an arbitrary choice of an orthogonal system in the space \( (x_1, \ldots, x_n) \), we have

\[
\lim_{\delta \to 0^+} \text{essinf} |\mathbf{z} - \mathbf{x}(s)| < \delta \{a(s, \mathbf{z})\} \leq \mathcal{X}'(s) \leq \lim_{\delta \to 0^+} \text{esssup} |\mathbf{z} - \mathbf{x}(s)| < \delta \{a(s, \mathbf{z})\},
\]

where \( a \) is the right-hand side of the system (5) in the chosen orthogonal system of coordinates.

With this notation we have the following theorem (see Theorem 2.1 in [15]).

**Theorem 2.** Let \( \mathbf{a} \) satisfy (4). Then for every \( (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^N \) there exist at least one Filippov characteristic \( \mathcal{X}(s) := \mathcal{X}(s; (t, \mathbf{z})) \). Moreover, we have

(i) if these characteristics are unique (on the right) for \( 0 \leq t \leq s \leq T \) then the flow, i.e., the map \( (s, \mathbf{z}) \to \mathcal{X}(s; (t, \mathbf{z})) \) is continuous on \([t, T] \times \mathbb{R}^N \),

(ii) if these characteristics are unique (on the left) for \( 0 \leq s \leq t \leq T \) then the map \( (s, \mathbf{z}) \to \mathcal{X}(s; (t, \mathbf{z})) \) is continuous on \([0, t] \times \mathbb{R}^N \).

Filippov has given the following uniqueness criterion: let \( L(t, z) \) be a function in \( L^1([0, T]; L^\infty(\mathbb{R})) \) such that for any \( 0 \leq s_0 \leq T \) the only solution to the scalar equation

\[
z'(s) = L(s, z(s)) \quad \text{a.e. } s_0 \leq s \leq T, \quad z(s_0) = 0,
\]

is identically equal to 0. Then we have
Theorem 3. Let $L$ satisfy the above uniqueness property.

(i) If $\mathbf{a}$ satisfies
\[
(a(t, \mathbf{z}) - a(t, \mathbf{y})) \cdot (\mathbf{z} - \mathbf{y}) \leq |\mathbf{z} - \mathbf{y}|L(t, |\mathbf{z} - \mathbf{y}|), \text{ a.e } (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^N,
\] then the characteristics are unique on the right. That is, if $\chi^1(\cdot; (t, \mathbf{z}))$ and $\chi^2(\cdot; (t, \mathbf{z}))$ are two Filippov characteristics passing through $\mathbf{z}$ at time $t$, then $\chi^1(s; (t, \mathbf{z})) = \chi^2(s; (t, \mathbf{z}))$ for $t \leq s \leq T$.

(ii) If $\mathbf{a}$ satisfies
\[
(a(t, \mathbf{z}) - a(t, \mathbf{y})) \cdot (\mathbf{z} - \mathbf{y}) \geq -|\mathbf{z} - \mathbf{y}|L(T-t, |\mathbf{z} - \mathbf{y}|), \text{ a.e } (t, \mathbf{z}) \in [0, T] \times \mathbb{R}^N,
\] then the characteristics are unique on the left. That is, if $\chi^1(\cdot; (t, \mathbf{z}))$ and $\chi^2(\cdot; (t, \mathbf{z}))$ are two Filippov characteristics passing through $\mathbf{z}$ at time $t$, then $\chi^1(s; (t, \mathbf{z})) = \chi^2(s; (t, \mathbf{z}))$ for $0 \leq s \leq t$.

Remark 1. There are many possibilities for $L$. For example, see [9], we can take $L(t, z) = m(t)\phi(z)$ with $m \in L^1[0, T]$, $\phi$ continuous and positive for $z > 0$ and $\int_0^\tau d\zeta/\phi(\zeta) = +\infty$ for any $\tau \in (0, T]$. The special case $\phi(z) = z$ is used in [1,15,17].

We introduce the following smoothing procedure. Let $\rho$ be a $C^\infty$ non-negative function, supported on $B(0, 1) := \{ \mathbf{z} : |\mathbf{z}| \leq 1 \}$, $\int_{\mathbb{R}^N} \rho(\mathbf{z})d\mathbf{z} = 1$, and denote by $\rho^\delta(\mathbf{z}) := \delta^{-N}\rho(\mathbf{z}/\delta)$. We define
\[
a^n := (-n \wedge n) \ast \rho^\frac{1}{\delta},
\] i.e., $a^n$ is the smooth cutoff of the function $a$. Let $\chi^n$ be the solution of the smooth (with respect to $\mathbf{z}$) system
\[
\frac{d\chi^n}{ds} = a^n(s, \chi^n), \quad 0 \leq s \leq T,
\] for $n = 1, 2, \ldots$. The following theorem is a corollary of Theorem 11 in [9].

Theorem 4. Suppose that the transport velocity $\mathbf{a}$ satisfies (4).

(i) Assume that $\mathbf{a}$ is such that the Filippov characteristics are unique on the right. Let $\chi^n(\cdot) = \chi^n(\cdot; (t, \mathbf{z}))$ be a sequence of solutions to (10) such that $\chi^n(t^n; (t^n, \mathbf{z}^n)) = \mathbf{z}^n$ where $t^n \to t$ and $\mathbf{z}^n \to \mathbf{z}$ as $n \to \infty$. Then $\chi^n(\cdot)$ converges uniformly to $\chi(\cdot; (t, \mathbf{z}))$ on the interval $[t, T]$.

(ii) We have the corresponding result in the case the Filippov characteristics are unique on the left. Let $\chi^n(\cdot) = \chi^n(\cdot; (t, \mathbf{z}))$ be a sequence of solutions to (10) such that $\chi^n(t^n; (t^n, \mathbf{z}^n)) = \mathbf{z}^n$ where $t^n \to t$ and $\mathbf{z}^n \to \mathbf{z}$ as $n \to \infty$. Then $\chi^n(\cdot)$ converges uniformly to $\chi(\cdot; (t, \mathbf{z}))$ on the interval $[0, t]$.

Remark 2. The theorem says that once we have one-sided uniqueness (left or right) of the characteristics, we get continuous dependence on the initial condition $(t, \mathbf{z})$ and the mollification (9) of $\mathbf{a}$. For example, we have uniqueness on the right if $\mathbf{a}$ satisfies (7), and we have uniqueness on the left if $\mathbf{a}$ satisfies (8). There are other ways to get one-sided uniqueness of the Filippov characteristics, see the notion of $\mu$-monotonicity introduced in [13].

We will use the notion of continuous convergence, introduced by Carathéodory.
Definition 5. Let $X$ and $Y$ be metric spaces. A sequence of maps $\{f^n\}$ from $X$ to $Y$ is said to converge continuously at a point $z$ to a map $f$ of $X$ into $Y$ if, for each sequence $\{z^n\} \subset X$, with $z = \lim z^n$, we have $f(z) = \lim f^n(z^n)$. We say that $f^n$ converges continuously to $f$ on $X$ if $f^n$ converges continuously to $f$ at each $z \in X$.

The following theorem is a corollary from Lemma 28-9-4 and Lemma 28-9-51 in [10].

Theorem 6. A sequence $\{f^n\}$ of continuous maps of $X$ into $Y$ converges continuously to $f$ on $X$ if and only if $\{f^n\}$ converges uniformly to $f$ on each compact subset of $X$.

Now, we will show that the smooth flow $X^n(0; \cdot)$ converges uniformly to the Filippov flow $X(0; \cdot)$ on compact sets.

Theorem 7. Let $a$ be such that the Filippov characteristics are unique on the left. Then $X^n(0, \cdot)$ defined by (10) converges uniformly to $X(0; \cdot)$ on each compact subset of $[0, T] \times \mathbb{R}^N$.

Proof: The Filippov flow $X(0; \cdot)$ is well defined on $[0, T] \times \mathbb{R}^N$ because the Filippov characteristics are unique on the left. We have that $X^n(0; \cdot) : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ for all $n$. Let $z^n := (t^n, x^n)$ be a sequence of points converging to $z := (t, x)$. By Theorem 4-(ii), we have $X^n(0; z^n) \rightarrow X(0; z)$. Hence $X^n(0; \cdot)$ converges continuously to $X(0; \cdot)$ on $[0, T] \times \mathbb{R}^N$. Using Theorem 6, we conclude that $X^n(0; \cdot)$ converges uniformly to $X(0; \cdot)$ on each compact subset of $[0, T] \times \mathbb{R}^N$. □

§3. Weak solutions

We want to study the relation between the Filippov flow, introduced in the previous section, and weak solutions to (1). We are going to show that if the Filippov flow is unique, then there exist a unique weak solution which is stable under the smoothing procedure (9) of the the transport velocity $a$. We also give a formula for the stable solution using the Filippov flow, see Theorem 11.

We start with a classical result, see [2,3,16].

Theorem 8. Let $a \in (C; \text{Lip}1)$ and $D_i a_j \in C([0, T] \times \mathbb{R}^N)$ for all $1 \leq i, j \leq N$. Then (1) has a unique solution given by

$$u(t, x) = u^0(X(0; (t, x))). \quad (11)$$

Recall that $a^n := (-n \vee a \wedge n) * \rho^{1/n}$ for $n = 1, 2, \ldots$. We will use the following lemma.

Lemma 9. Let $K \subset \mathbb{R}^N$ be a compact set and $f^n$ be a sequence of continuous functions converging uniformly to $f$ on $[0, T] \times K$. Let $a$ satisfy (3) and (4), and $a^n$ be as above. Then the total mass of the measure $\text{div}_x(a^n)$ on $[0, T] \times K$ is bounded by an absolute constant $C$ for all $n \geq 1$ and

$$\int_0^T \int_K f_{\text{div}} a dx dt = \lim_{n \to \infty} \int_0^T \int_K f^n_{\text{div}} a^n dx dt. \quad (12)$$
Proof: We use (3) to prove that the total mass of $\text{div}_z(a^n)$ on $K$ is bounded by a constant that does not depend on $n$. We just sketch the proof of this fact. It is enough to prove this in the case when $K$ is a rectangular cell, $K = [x_1^0, y_1^0] \times [x_2^0, y_2^0] \times \cdots \times [x_N^0, y_N^0]$. Let’s denote $K =: [x_i^0, y_i^0] \times K'$, $(t, x_i, z') := (t, x, z')$ and $a_i(x_i) := a_i(t, x_i)$ where we suppress the $(t, z')$ dependence to simplify the notation. If $\mu$ is a Radon measure on $[0, T] \times \mathbb{R}^N$ and $P$ is a rectangular cell, then we denote the total mass (variation) of $\mu$ on $[0, T] \times P$ by $\|\mu\|([0, T] \times P)$.

We know that $D_i a_i$ is a Radon measure on $[0, T] \times \mathbb{R}^N$. By a result of Krickeberg (see Theorem 5.2 in [11]), we have that $D_i a_i$ is a Radon measure on $[0, T] \times \mathbb{R}^N$ if and only if $a_i$ has a representative (denoted again by $a_i$) such that the function $\text{Var}_{[x_i^0, y_i^0]} a_i$ as a function of $(t, z')$ is integrable on $[0, T] \times P'$ ($P' \subset \mathbb{R}^{N-1}$ is defined the same way $K'$ is), and the total mass of $D_i a_i$ on $P$ is 

$$\|D_i a_i\|([0, T] \times P) = \int_0^T \int_{P'} \text{Var}_{[x_i^0, y_i^0]} a_i \, dz' \, dt$$

for all rectangular cells $P \in [0, T] \times \mathbb{R}^N$ and all $1 \leq i \leq N$. Using standard properties of convolution, it is easy to show that

$$\|\text{div}_z a^n\|([0, T] \times K) \leq \|\text{div}_z (-n \lor a \land n)\|([0, T] \times B(K, 1))$$

where $B(K, 1) := [x_1^0 - 1, y_1^0 + 1] \times [x_2^0 - 1, y_2^0 + 1] \times \cdots \times [x_N^0 - 1, y_N^0 + 1]$. Using the above estimates, we derive

$$\|\text{div}_z a^n\|([0, T] \times K) \leq \|\text{div}_z (-n \lor a \land n)\|([0, T] \times B(K, 1)) \leq \sum_{i=1}^N \|D_i a_i\|([0, T] \times B(K, 1)) =: C.$$ 

Hence the total mass of $\text{div}_z(a^n)$ is bounded by a constant $C$ independent of $n$. This proves the first part of the lemma.

Let $g_1, g_2 \in C([0, T] \times K)$ and $\mu_1, \mu_2$ be Radon measures on $[0, T] \times \mathbb{R}^N$. We shall use the notation

$$|g_1 \mu_1 - g_2 \mu_2|_K := \left| \int_0^T \int_K g_1 \, d\mu_1 - \int_0^T \int_K g_2 \, d\mu_2 \right|.$$ 

For any integers $m, n \geq 1$, we have

$$|f^n \text{div}_z a^n - f \text{div}_z a^n|_K \leq |(f^n - f) \text{div}_z a^n|_K + |(f - f^m) \text{div}_z a^n|_K + |f^n (\text{div}_z a^n - \text{div}_z a)|_K + |(f^m - f) \text{div}_z a|_K,$$
and we derive the inequality

\[ |f^n \text{div}_z a^n - f \text{div}_z a|_K \leq \|f - f^n\|_{C([0,T] \times K)} \|\text{div}_z a^n\|([0,T] \times K) + \|f - f^n\|_{C([0,T] \times K)} \|\text{div}_z a\|([0,T] \times K) + \|\nabla_z f^m\|_{L^\infty([0,T] \times K)} \|a^n - a\|_{L^1([0,T] \times K)} + \|f^m - f\|_{C([0,T] \times K)} \|\text{div}_z a\|([0,T] \times K). \]

Hence,

\[ |f^n \text{div}_z a^n - f \text{div}_z a|_K \leq \|\nabla_z f^m\|_{L^\infty([0,T] \times K)} \|a^n - a\|_{L^1([0,T] \times K)} + C \left( \|f - f^n\|_{C([0,T] \times K)} + 2\|f^m - f\|_{C([0,T] \times K)} \right). \]

Using that \( \|f - f^n\|_{C([0,T] \times K)} \to 0 \), \( \|f^m - f\|_{C([0,T] \times K)} \to 0 \) and \( \|a^n - a\|_{L^1([0,T] \times K)} \to 0 \), we finish the proof of (12) by a standard \( \varepsilon-\delta \) argument (first we select \( m \) and then \( n \)). \( \square \)

We know, see [2,7,8,15,16], that the flow \( \mathcal{X} \) and the solution \( u \) to (1) are unique under the weaker assumptions \( a \in L^1([0,T];(C^1_b(\mathbb{R}^N))^N) \), provided that \( u^0 \in C^1_b(\mathbb{R}^N) \). The next theorem shows that the solution formula (11) still holds in this case.

**Theorem 10.** Let \( a \in L^1([0,T];(C^1_b(\mathbb{R}^N))^N) \) and \( u^0 \in C^1_b(\mathbb{R}^N) \). Then (1) has a unique weak solution which is given by (11).

**Proof:** The uniqueness follows by standard arguments, see [7] for details. So, we have to prove the representation formula (11). Let’s consider the modified problem

\[
\begin{cases}
  u_t + a^n \cdot \nabla_z u = 0, & (t,z) \in [0,T] \times \mathbb{R}^N, \\
  u(0,z) = u^0(z), & z \in \mathbb{R}^N,
\end{cases}
\]

where we use \( a^n := (-n \vee a \wedge n) * \rho^{1/n} \) for \( n = 1, 2, \ldots \). For the problem (13), we have that the representation formula \( u^n(t,\mathbf{z}) = u^0(\mathcal{X}^n(0);(t,\mathbf{z})) \) holds true, see [3] for details.

By Theorem 7, we have that \( \mathcal{X}^n(0;\cdot) \) converges uniformly to \( \mathcal{X}(0;\cdot) \) on each compact subset of \( [0,T] \times \mathbb{R}^N \). If \( K \subset [0,T] \times \mathbb{R}^N \) is compact then it is easy to show that \( \mathcal{X}^n(0;\cdot)(K) \) is inside a fixed compact set of \( \mathbb{R}^N \) for all \( n \). We know that \( u^0 \in C^1_b(\mathbb{R}^N) \), hence \( u^0 \) is uniformly continuous on each compact subset of \( \mathbb{R}^N \) and we derive that \( u^n = u^0(\mathcal{X}^n(0;\cdot)) \) converges uniformly to \( u^0(\mathcal{X}(0;\cdot)) \) by definition. We know that \( u^n \) is a classical solution and hence a weak solution to (1). Thus (2) is valid for \( u^n \) and \( a^n \). Fix a test function \( \phi \in C^\infty_0([0,T] \times \mathbb{R}^N) \). Then there exists a compact set \( K \subset [0,T] \times \mathbb{R}^N \) such that the support of \( \phi \) is inside \( K \). We have that \( u^n \) converges uniformly to \( u^0 \) on \( K \). We now take a limit in

\[
\int_0^T \int_{\mathbb{R}^N} u^n \phi_t + u^n \text{div}_z (a^n \phi) d\mathbf{z} dt = \int_{\mathbb{R}^N} u^0 (T,\mathbf{z}) \phi(T,\mathbf{z}) d\mathbf{z} - \int_{\mathbb{R}^N} u^0 \phi(0,\mathbf{z}) d\mathbf{z}
\]

and arrive at (2) provided that

\[
\int_0^T \int_{\mathbb{R}^N} u \text{div}_z (a \phi) d\mathbf{z} dt = \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^N} u^n \text{div}_z (a^n \phi) d\mathbf{z} dt. \tag{14}
\]
It is easy to observe that (14) holds if and only if
\[
\int_0^T \int_{\mathbb{R}^N} u_\phi \cdot \text{div}_x a \, dx \, dt = \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^N} u^n_\phi \cdot \text{div}_x a^n \, dx \, dt.
\]
and we finish the proof using Lemma 9 with \( f := u_\phi \) and \( f^n := u^n_\phi \). \( \square \)

The following theorem is our main result.

**Theorem 11.** Let \( a \) satisfy (3), (4) and be such that the Filippov flow is unique on the left. Then for any \( u^0 \in C(\mathbb{R}^N) \), there exists a continuous weak solution to (1) of the form (11). Moreover, this solution is the only weak solution to (1) which is stable under the smoothing (9) of the transport velocity \( a \) and the initial condition \( u^0 \).

**Proof:** We consider the smooth (with respect to \( z \)) problem
\[
\begin{cases}
  u_t + a^n \cdot \nabla u = 0, & (t, z) \in [0, T] \times \mathbb{R}^N, \\
  u(0, z) = u^{0,n}(z), & z \in \mathbb{R}^N,
\end{cases}
\]
where \( a^n := (-n \land a \land n) * \rho^{1/n} \) and \( u^{0,n} := (-n \land u^0 \land n) * \rho^{1/n} \). By Theorem 10 there exists a unique solution \( u^n \) to (16) given by
\[
u^n(t, z) = u^{0,n}(\lambda^n(0; (t, z))),
\]
where \( \lambda^n \) is the unique Filippov flow associated with \( \frac{d\lambda}{ds} = a^n(s, \lambda) \), with \( \lambda(t; (t, z)) = z \).

In order to prove Theorem 11, it is enough to show that the sequence \( u^n \) converges uniformly to \( u \) on each compact subset \( K \) of \( [0, T] \times \mathbb{R}^N \) and the total mass of \( \text{div}_x(a^n) \) on \( K \) is bounded for all \( n \). This is the case, because Lemma 9 still holds in the case \( \text{div}_x(a^n) \) and \( \text{div}_x(a) \) are only Radon measures and we use it to prove that \( u \) is a weak solution to (1). The representation formula (11) follows from the uniqueness and stability of the Filippov flow. Similar to Theorem 10, we use Theorem 7 to observe that \( u^n \) converges uniformly to \( u \) on each compact subset of \( [0, T] \times \mathbb{R}^N \). Note that we do not need that \( a \) satisfy (3) in order to prove the uniform convergence. We need (3) to prove that for any fixed compact set \( K \subset [0, T] \times \mathbb{R}^N \) the total mass of \( \text{div}_x(a^n) \) on \( K \) is bounded by a constant that does not depend on \( n \). We just sketch the proof of this fact. It is enough to prove this in the case when \( K \) is a rectangular cell, \( K = [0, T] \times [x^0_1, y^0_1] \times [x^0_2, y^0_2] \times \cdots [x^0_N, y^0_N] \). Let's denote \( K' := [x^0_i, y^0_i] \times K' \), \( (x_i, z') := (t, z) \), and \( a_i(x_i) := a_i(t, z) \). If \( \mu \) is a Radon measure on \( [0, T] \times \mathbb{R}^N \), then we denote the total mass (variation) of \( \mu \) on \( K \) by \( ||\mu||(K) \). We know that \( D_i a_i \) is a locally bounded measure on each compact subset of \( [0, T] \times \mathbb{R}^N \). By a result of Krickeberg (see Theorem 5.2 in [11]), we have that \( D_i a_i \) is a Radon measure on \( [0, T] \times \mathbb{R}^N \) if and only if \( a_i \) has a representative (denoted again by \( a_i \)) such that the function \( \text{Var}_{[x^0_i, y^0_i]} a_i \) as a function of \( z' \) is integrable on \( K' \) and the total mass \( ||D_i a_i||(K) = \int_{K'} \text{Var}_{[x^0_i, y^0_i]} a_i \, dz' \) for all rectangular cells \( K \in [0, T] \times \mathbb{R}^N \).
It is easy to show that the cutoff function \(-n \vee a_i \wedge n\) does not increase the variation in direction \(x_i\) for all \(1 \leq i \leq N\). That is, \(\text{Var}_{[x_i^0, y_i^0]}(-n \vee a_i \wedge n) \leq \text{Var}_{[x_i^0, y_i^0]} a_i\), for all \(1 \leq i \leq N\). Hence
\[
\|D_i(-n \vee a_i \wedge n)\|((K) \leq \|D_i a_i\|((K)
\]
for all rectangular cells \(K \in [0, T] \times \mathbb{R}^N\) and all \(1 \leq i \leq N\). Using standard properties of convolution, it is easy to show that
\[
\|\text{div}_x a^n\|((K) \leq \|\text{div}_x (-n \vee a \wedge n)\|((K + 1)
\]
where \(K + 1 := [0, T] \times [x_1^0 - 1, y_1^0 + 1] \times [x_2^0 - 1, y_2^0 + 1] \times \cdots \times [x_N^0 - 1, y_N^0 + 1]\).

Using the above estimates, we derive
\[
\|\text{div}_x a^n\|((K) \leq \|\text{div}_x (-n \vee a \wedge n)\|((K + 1) \leq \sum_{i=1}^{N} \|D_i a_i\|((K + 1).
\]

Hence the total mass of \(\text{div}_x (a^n)\) is bounded by a constant independent of \(n\). Using Lemma 9 in this setting, we conclude that \(u\) satisfies (2), i.e., \(u\) is a weak solution. We know that \(u^n\) converges to \(u\) for any smoothing of \(a\) and \(a^0\) of type (9), therefore the uniqueness of a stable solution follows from the uniqueness and stability of the Filippov flow (see Theorem 4 and Theorem 7). \(\square\)

References


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