Math142 Lecture Notes

1.7 - Composite Functions & Logarithmic Functions

Suppose that an oil refinery’s underwater supply line ruptures, resulting in an oil spill that is fairly circular. Local companies concerned about the environment would like to know how much area the spill will cover as time passes. Results from a similar disaster show that the radius of the spill increases at a rate of 0.7 feet per second. Find a formula that will give the area of the spill as a function of time.

If we let \( r = \) radius of the spill in feet, then the area of the spill is defined by

\[
A = \pi r^2
\]

The spill is growing at a rate express in time, where \( t = \) the time in seconds:

\[
r = 0.7t
\]

To find the total area of the spill as a function of time, find:

\[
A \circ r = A(r(t)) = \pi (0.7t)^2
\]

This then gives the area of the spill as a function of time:

\[
A(t) = 0.49\pi t^2
\]

Composite Function

A function \( h \) is a composition of the functions \( f \) and \( g \) if

\[
h(x) = f(g(x)) \\
f(g(x)) = (f \circ g)(x)
\]

Example 1: Let \( f(x) = 6x - 7 \), \( g(x) = x^2 + 3 \), and \( h(x) = \sqrt{4x - 12} \).

(a) Find \( g(f(2)) = (g \circ f)(2) = \)

(b) Find \( f(g(2)) = (f \circ g)(2) = \)

(c) Find \( f(g(x)) = (f \circ g)(x) = \) and its domain.

(d) Find \( h(f(x)) = (h \circ f)(x) = \) and its domain.
Example 2: Determine functions $f$ and $g$ such that $h(x) = f(g(x))$ if

(a) $h(x) = \left(\frac{5}{x - 1}\right)^4$  
(b) $h(x) = \sqrt{x + 5}$

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**One-to-One Function**

A function $f$ is a one-to-one function if, for elements $a$ and $b$ in the domain of $f$, $a \neq b$ implies that $f(a) \neq f(b)$

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Example 3: Determine if the following functions are one-to-one.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
</tr>
</tbody>
</table>

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Two functions are inverses of each other if:

1. $g(f(x)) = x$ and $f(g(x)) = x$.
2. their $x$ and $y$ values are interchanged.
3. their graphs are symmetrical about the line $y = x$

The notation $f^{-1}(x) = g(x)$ is read “the inverse of $f(x)$ is $g(x)$”.

Example 4: For the one-to-one functions $f(x) = \frac{x - 3}{4}$ and $g(x) = 4x + 3$, show that $g$ and $f$ are inverses.
Logarithm

\[ y = \log_b x \text{ if and only if } b^y = x \]

where \( b > 0, b \neq 1, x > 0 \)

*(\( \log_b x \) is read "log base \( b \) of \( x \)"")

The following diagram is helpful when converting from a logarithm to an exponential.

\[ \log_b x = y \quad \Leftrightarrow \quad b^y = x \]

"\( b \) to the \( y \) equals \( x \"")

A similar diagram is helpful when converting from an exponential to a logarithm.

\[ b^y = x \quad \Leftrightarrow \quad \log_b x = y \]

"\( \log \) base \( b \) to of \( x \) equals \( y \"")

There are two special logarithms that are frequently used and have a special notation. One is called a common logarithm (\( \log_{10} \)) and the other is called the natural logarithm (\( \log_e \)). They are typically written as "log" and "ln" respectively.

Example 5: Rewrite each exponential equation in logarithmic form.
(a) \( 3^4 = 81 \)  
(b) \( e^0 = 1 \)  
(c) \( 10^{-2} = 0.01 \)

Example 6: Rewrite each logarithmic equation in exponential form.
(a) \( \log 100 = 2 \)  
(b) \( \ln 20.1 = 3 \)  
(c) \( \log_3 81 = 4 \)

Properties of Logarithms

Let \( m \) and \( n \) be positive numbers. Then the following are true:

1. \( \log_b (mn) = \log_b m + \log_b n \)
2. \( \log_b \left( \frac{m}{n} \right) = \log_b m - \log_b n \)
3. \( \log_b m^n = n \log_b m \)
4. \( \log_b m = \log_b n \) if and only if \( m = n \)
5. \( \log_b b = 1 \)
6. \( b^{\log_b x} = x \)
7. \( \log_b b^x = x \)
8. \( \log_b 1 = 0 \)
Example 7: Given \( \log_b 5 = 2.3220 \) and \( \log_b 3 = 1.5840 \) find the following.

(a) \( \log_b 75 \)

(b) \( \log_b \left( \frac{9}{5} \right) \)

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**Evaluating Logarithms in the Calculator**

1) If it is a common logarithm (\( \log_{10} \)), use the \[ \log \] button.

2) If it is a natural logarithm (\( \log_e \)), use the \[ \ln \] button.

3) If it is not a common or natural logarithm, use the change of base formula.

\[
\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}
\]

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Example 8: Solve the following equations. Find (i) the **exact** solution, and (ii) an approximate solution rounded to four decimal places.

(a) \( 2^x = 10 \)

(b) \( 4e^x = 81 \)

(c) \( 0.33(1.27)^x = 58 \)