2.2 - Limits and Asymptotes

Consider the graph of \( f(x) = \frac{1}{x} \) below.

\[
\lim_{x \to 0^+} f(x) = \quad \lim_{x \to 0^-} f(x) = \quad \text{therefore} \quad \lim_{x \to 0} f(x) =
\]

Consider the graph of \( g(x) = \frac{1}{x^2} \) below.

\[
\lim_{x \to 0^+} g(x) = \quad \lim_{x \to 0^-} g(x) = \quad \text{therefore} \quad \lim_{x \to 0} g(x) =
\]

**Infinite Limits and Vertical Asymptotes**

The line \( x = a \) is a vertical asymptote of the graph of \( f \) if any of the following are true.

- \( \lim_{x \to a^-} f(x) = \infty \) or \( \lim_{x \to a^-} f(x) = -\infty \).
- \( \lim_{x \to a^+} f(x) = \infty \) or \( \lim_{x \to a^+} f(x) = -\infty \).
Example 1: Find (a) \( \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \) \hspace{2cm} (b) \( \lim_{x \to 1} \frac{x - 1}{x^2 - 1} \)

Example 2: The cost, \( C(x) \), in thousands of dollars of removing \( x\% \) of a city’s pollutants discharged into a lake is given by

\[
C(x) = \frac{113x}{100 - x}
\]

(a) Determine a reasonable domain for \( C \).

(b) Evaluate \( C(50) \) and interpret.

(c) Determine \( \lim_{x \to 100^-} C(x) \) and interpret.
Examine the graphs of \( f(x) = e^x \) and \( g(x) = e^{-x} \) below.

Find
\[
\lim_{x \to \infty} f(x) =
\]
\[
\lim_{x \to -\infty} f(x) =
\]

Find
\[
\lim_{x \to \infty} g(x) =
\]
\[
\lim_{x \to -\infty} g(x) =
\]

**Limits at Infinity and Horizontal Asymptotes**

For any function \( f \), if \( \lim_{x \to \pm \infty} f(x) = L \), then the line \( y = L \) is a horizontal asymptote for the graph of \( f \).

**Special Limits at Infinity**

1. For \( n \), a positive real number, \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \)

2. For \( n \), a positive real number, \( \lim_{x \to -\infty} \frac{1}{x^n} = 0 \), provided that \( x^n \) is a real number for negative values of \( x \).

Example 3: Find the following.

(a) \( \lim_{x \to \infty} \frac{2x + 5}{x - 1} \)

(b) \( \lim_{x \to \infty} \frac{3x^2 - 2x + 5}{2x^3 + x^2 - 2x + 3} \)
Definition Average Cost

If $C(x)$ is a cost function, then the average cost is given by

$$AC(x) = \overline{C}(x) = \frac{C(x)}{x}$$

Example 4: The total cost, in dollars, to produce $x$ units of a certain product is given by $C(x) = 22,500 + 7.35x$. Find $\lim_{x \to \infty} \overline{C}(x)$ and interpret.

**Horizontal Asymptotes**

Remember: horizontal asymptotes can be found by comparing the degree of the numerator and denominator if they are polynomials, or by dividing each term by the highest power in the denominator.

If the degree of the numerator is larger than the degree of the denominator, then no horizontal asymptote exist, AND the limit of the function as $x \to \infty$ is either $\pm \infty$.

If the degree of the denominator is larger than the degree of the numerator, then $y = 0$ is the horizontal asymptote, AND the limit of the function as $x \to \infty$ is zero.

If the degree of the numerator and denominator are the same, the horizontal asymptote is found by dividing the leading coefficients, AND the limit of the function as $x \to \infty$ is $\frac{a}{b}$.

Example 5: Find the following limits.

(a) $\lim_{x \to \infty} \frac{e^x - e^{-x}}{2e^x + 3e^{-x}}$

(b) $\lim_{x \to \infty} \frac{4e^x - e^{-x}}{3e^{-x} + 5e^x}$

(c) $\lim_{x \to -\infty} \frac{e^x - 3e^{-x}}{2e^x + 7e^{-x}}$

(d) $\lim_{x \to -\infty} \frac{4e^x - 5e^{-x}}{3e^{-x} + 4e^x}$