Homework Solutions Section 3.2

> with (linalg):

1 Problem 1

1.1 Problem 1a)  
((x1,x2)+(y1,y2)=(x1+y1,x2+y2). Since (x1+y1)+(x2+y2)=(x1+x2)+(y1+y2)=0+0=0, it is closed under addition. Since k*(x1+k*x2)=k*(x1+x2)=k*0=0, (k*x1,k*x2) is in the space, it is closed under scalar multiplication.

1.2 Problem 1b)  
Since (k*x1)*(k*x2)=k*k*x1*x2=k*k*0=0, it is closed under scalar multiplication. ((x1+y1)*(x2+y2)=x1*x2+x1*y2+y1*x2+y1*y2 =x1*y2+y1*x2  
This will not vanish for arbitrary values of x1,x2,y1,y2! Space is not closed under scalar multiplication.

1.3 Problem 1c)  
((x1+y1,x2+y2) : (x1+y1) = (3*x2+3*y2)=3*(x2+y2) So it is closed under addition. k(x1,x2): (k*x1)=(k*3*x2)=3*(k*x2) So it closed under scalar multiplication. It is a subspace.

1.4 Problem 1d)  
k*(x1,x2)=(k*x1,k*x2). —k*x1—=—k—*—x1—=—k—*—x2—=—k*x2— So it is closed under scalar multiplication. ((x1+y1)’2=(x2+y2)’2, or x1’2 + 2*x1*y1+y1’2 =? x2’2 + 2*x2*y2+y2’2, true if and only if x1*y1 = x2*y2. This is not true in general, take (x1,x2) = (2,2) and (y1,y2)=(2,-2). These are both in the space, but (2,2)+(2,-2)=(4,0) is not!

1.5 Problem 1e)  
This is closed under scalar multiplication, since (k*x1)”2 = k’2 * x1”2 = k’2 * x2”2 = (k*x2)”2. However, it is not closed under addition. Take for example the vectors (1,1) and (1,-1). These are both in the subset, but (1,1)+(1,-1)=(2,0) is not!

2 Problem 3

2.1 Problem 3a  
> diag1 := array(1..2,1..2,[[d1,0],[0,d2]]):
> diag2 := array(1..2,1..2,[[e1,0],[0,e2]]):

```
\[
diag1 := \begin{bmatrix} d1 & 0 \\ 0 & d2 \end{bmatrix} \\
\]
\[
diag2 := \begin{bmatrix} e1 & 0 \\ 0 & e2 \end{bmatrix} \\
\]

2.1.1 closed under addition
> matadd(diag1,diag2);
\[
\begin{bmatrix} d1 + e1 & 0 \\ 0 & d2 + e2 \end{bmatrix}
\]

2.1.2 closed under scalar multiplication
> scalarmul(diag1,k);
\[
\begin{bmatrix} k d1 & 0 \\ 0 & k d2 \end{bmatrix}
\]

2.2 Problem 3b
> tri1 := array(1..2,1..2,[[a1,a2],[0,a3]]);
> tri2 := array(1..2,1..2,[[b1,b2],[0,b3]]);
\[
\begin{bmatrix} a1 & a2 \\ 0 & a3 \end{bmatrix} \\
\begin{bmatrix} b1 & b2 \\ 0 & b3 \end{bmatrix}
\]

2.2.1 closed under addition
> matadd(tri1,tri2);
\[
\begin{bmatrix} a1 + b1 & a2 + b2 \\ 0 & a3 + b3 \end{bmatrix}
\]

2.2.2 closed under scalar multiplication
> scalarmul(tri1,k);
\[
\begin{bmatrix} k a1 & k a2 \\ 0 & k a3 \end{bmatrix}
\]

2.3 Problem 3c
> mat1 := array(1..2,1..2,[[a1,1],[a2,a3]]);
> mat2 := array(1..2,1..2,[[b1,1],[b2,b3]]);
\[
\begin{bmatrix} a1 & 1 \\ a2 & a3 \end{bmatrix} \\
\begin{bmatrix} b1 & 1 \\ b2 & b3 \end{bmatrix}
\]

2.3.1 Not closed under addition
> matadd(mat1,mat2);
\[
\begin{bmatrix} a1 + b1 & 2 \\ a2 + b2 & a3 + b3 \end{bmatrix}
\]

2.4 Problem 3d
> mat1 := array(1..2,1..2,[[0,a1],[a2,a3]]);
> mat2 := array(1..2,1..2,[[0,b1],[b2,b3]]);
\[
\begin{bmatrix} 0 & a1 \\ a2 & a3 \end{bmatrix} \\
\begin{bmatrix} 0 & b1 \\ b2 & b3 \end{bmatrix}
\]
\[
mat1 := \begin{bmatrix}
0 & a1 \\
a2 & a3 
\end{bmatrix}
\]
\[
mat2 := \begin{bmatrix}
0 & b1 \\
b2 & b3 
\end{bmatrix}
\]

2.4.1 closed under addition
\[
> \text{matadd(mat1,mat2)};
\]
\[
\begin{bmatrix}
0 & a1 + b1 \\
a2 + b2 & a3 + b3 
\end{bmatrix}
\]

2.4.2 closed under scalar multiplication
\[
> \text{scalarmul(mat1,k)};
\]
\[
\begin{bmatrix}
0 & k a1 \\
k a2 & k a3 
\end{bmatrix}
\]

2.5 Problem 3e
\[
> \text{mat1 := array(1..2,1..2,[[a1,a2],[a2,a3]]);} \\
> \text{mat2 := array(1..2,1..2,[[b1,b2],[b2,b3]]);} \\
\]
\[
mat1 := \begin{bmatrix}
a1 & a2 \\
a2 & a3 
\end{bmatrix}
\]
\[
mat2 := \begin{bmatrix}
b1 & b2 \\
b2 & b3 
\end{bmatrix}
\]

2.5.1 closed under addition
\[
> \text{matadd(mat1,mat2)};
\]
\[
\begin{bmatrix}
a1 + b1 & a2 + b2 \\
a2 + b2 & a3 + b3 
\end{bmatrix}
\]

2.5.2 closed under scalar multiplication
\[
> \text{scalarmul(mat1,k)};
\]
\[
\begin{bmatrix}
k a1 & k a2 \\
k a2 & k a3 
\end{bmatrix}
\]

2.6 Problem 3f
\[
> \text{mat1 := array(1..2,1..2,[[1,0],[0,0]]);} \\
> \text{mat2 := array(1..2,1..2,[[0,0],[0,1]]);} \\
\]
\[
mat1 := \begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
\]
\[
mat2 := \begin{bmatrix}
0 & 0 \\
0 & 1 
\end{bmatrix}
\]

2.6.1 mat1, mat2 are both singular
\[
> \text{det(mat1)};
\]
0
\[
> \text{det(mat2)};
\]
0
2.6.2 but the sum is the identity, which is not singular! Space is not closed under addition

```
> matadd(mat1,mat2);

```

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

3 Problem 4

3.1 Problem 4a)

```
> A := array(1..2,1..2,[[2,1],[3,2]]);

A :=
\[
\begin{bmatrix}
2 & 1 \\
3 & 2
\end{bmatrix}
\]

> x := array(1..2,1..1,[[a],[b]]);

x :=
\[
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]

> y := multiply(A,x);

y :=
\[
\begin{bmatrix}
2 a + b \\
3 a + 2 b
\end{bmatrix}
\]

> eq1 := y[1,1];

\[
eq 1 := 2 a + b
\]

> eq2 := y[2,1];

\[
eq 2 := 3 a + 2 b
\]

> solve({eq1,eq2},{a,b});

\[
\{a = 0, b = 0\}
\]

nullspace contains only (0,0)

3.2 Problem 4b)

```
> A := array(1..2,1..4,[[1,2,-3,-1],[-2,-4,6,3]]);

A :=
\[
\begin{bmatrix}
1 & 2 & -3 & -1 \\
-2 & -4 & 6 & 3
\end{bmatrix}
\]

> x := array(1..4,1..1,[[a],[b],[c],[d]]);

x :=
\[
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\]

> y := multiply(A,x);

y :=
\[
\begin{bmatrix}
a + 2 b - 3 c - d \\
-2 a - 4 b + 6 c + 3 d
\end{bmatrix}
\]

> eq1 := y[1,1];

\[
eq 1 := a + 2 b - 3 c - d
\]
eq2 := y[2,1];

\[
\text{eq2} := -2 a - 4 b + 6 c + 3 d
\]

\[
\text{solve(}\{\text{eq1,eq2}\},\{a,b,c,d\})
\]

\[
\{d = 0, b = b, c = c, a = -2 b + 3 c\}
\]

\[
(a,b,c,d) = (-2b + 3c, b, c, 0) = b(-2,1,0,0) + c(3,0,1,0)
\]

3.3 Problem 4c)

> A := array(1..3,1..3,[[1,3,-4],[2,-1,-1],[-1,-3,4]]);

\[
A := \begin{bmatrix}
1 & 3 & -4 \\
2 & -1 & -1 \\
-1 & -3 & 4 \\
\end{bmatrix}
\]

> x := array(1..3,1..1,[[a],[b],[c]]);

\[
x := \begin{bmatrix}
a \\
b \\
c \\
\end{bmatrix}
\]

> y := multiply(A,x);

\[
y := \begin{bmatrix}
a + 3b - 4c \\
2a - b - c \\
-a - 3b + 4c \\
\end{bmatrix}
\]

> eq1 := y[1,1];

\[
\text{eq1} := a + 3b - 4c
\]

> eq2 := y[2,1];

\[
\text{eq2} := 2a - b - c
\]

> eq3 := y[3,1];

\[
\text{eq3} := -a - 3b + 4c
\]

> solve({eq1,eq2,eq3},{a,b,c});

\[
\{b = c, a = c, c = c\}
\]

\[
(c,c,c) = c(1,1,1)
\]

3.4 Problem 4d)

> A := array(1..3,1..4,[[1,1,-1,2],[2,2,-3,1],[-1,-1,0,-5]]);

\[
A := \begin{bmatrix}
1 & 1 & -1 & 2 \\
2 & 2 & -3 & 1 \\
-1 & -1 & 0 & -5 \\
\end{bmatrix}
\]

> x := array(1..4,1..1,[[a],[b],[c],[d]]);

\[
x := \begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix}
\]
> y := multiply(A,x);
> eq1 := y[1,1];
> eq2 := y[2,1];
> eq3 := y[3,1];
> solve({eq1,eq2,eq3},{a,b,c,d});

(a,b,c,d) = (-b-5d,b,-3d,d) = b(-1,1,0,0)+d(-5,0,-3,1)

4 Problem 6

4.1 Problem 6a)
S = { f — f continuous, and f(1) = f(-1) }
If f and g are in S, (f+g)(1)=f(1)+g(1)=f(-1)+g(-1)=(f+g)(-1) so it is closed under addition.
k*f(1)=k*f(-1)=(kf)(-1), so it is closed under scalar multiplication. It is a subspace!

4.2 Problem 6b)
S = Set of odd functions = { f — f(-x)=-f(x) }
If f and g are in S, (f+g)(-x)=f(-x)+g(-x)=-f(x)-g(x)=-(f+g)(x)
so it is closed under addition.
(k*f)(-x)=k*f(-x)=k*(-f(x))=-k*f(x)=-(k*f)(x)
so it closed under scalar multiplication. It is a subspace!

4.3 Problem 6c)
S = { f — f is continuous, f is nondecreasing on [0,1] } f(x) = x is in S, but -f(x)=-x is not! Set is not closed under scalar multiplication. Not a subspace.

4.4 Problem 6d)
S = { f — f is continuous, and f(-1)=0=f(1) }
(f+g)(-1)=f(-1)+g(-1)=0+0=0 (f+g)(1)=f(1)+g(1)=0+0=0 So set is closed under addition.
(k*f)(-1)=k*f(-1)=k*0=0
(k*f)(1)=k*f(1)=k*0=0
So set is closed under scalar multiplication.
S is a subspace!

4.5 Problem 6e)
S = \{ f \mid f \text{ is continuous and } f(-1)=0 \text{ OR } f(1) = 0 \}
f = 1+x \text{ is in } S \text{ and } g = 1-x \text{ is in } S. \text{ However,}
the sum } f+g=2 \text{ is not in } S. \text{ Hence, } S \text{ is NOT}
a subspace!

5 Problem 10
5.1 Problem 10a)
> A := array(1..3,1..3,[[1,0,1],[0,1,0],[0,1,1]]);

\[
A := \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\]
> det(A);

1
A is non singular, so we can express any vector x as [A]c, and solve for the
coefficients c.
It therefore spans.

5.2 Problem 10b)
> A := array(1..3,1..4,[[1,0,1,1],[0,1,0,2],[0,1,1,3]]);

\[
A := \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 1 & 1 & 3 \\
\end{bmatrix}
\]
Find the echelon form of A
> gausselim(A);

\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
Ac=x will therefore have a solution (more than one, actually), so the set of
vectors spans. The set of vectors spans.

5.3 Problem 10c)
> A := array(1..3,1..3,[[2,3,2],[1,2,2],[-2,-2,0]]);
5.4 Problem 10d)

\[ A := \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{bmatrix} \]

\[ \text{det}(A); \]

0

A is singular, so you cannot solve \( Ac = x \) for arbitrary \( x \). Therefore vectors does not span!

5.5 Problem 10e)

\[ A := \begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \]

\[ \text{det}(A); \]

0

A is singular, so you cannot solve \( Ac = x \) for arbitrary \( x \). Therefore vectors does not span!

6 Problem 11

6.1 Problem 11a)

\[ x1 := \text{array}(1..3,1..1,[[[-1],[2],[3]]]); \]
\[ x2 := \text{array}(1..3,1..1,[[[3],[4],[2]]]); \]
\[ xvec := \text{array}(1..3,1..1,[[[2],[6],[6]]]); \]
\[ yvec := \text{array}(1..3,1..1,[[[-9],[-2],[6]]]); \]

\[ xI := \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \]
\[ x_2 := \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \]
\[ xvec := \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix} \]
\[ yvec := \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix} \]

> scalarmul(x1,c1)+scalarmul(x2,c2);
\[
\begin{bmatrix}
-c1 \\
2 c1 \\
3 c1
\end{bmatrix} +
\begin{bmatrix}
3 c2 \\
4 c2 \\
2 c2
\end{bmatrix}
\]

> solve(\{-c1+3*c2=2,2*c1+4*c2=6,3*c1+2*c2=6\},\{c1,c2\});
This does not return a solution, hence not in span.

6.1.1 Alternately, look at augmented matrix reduced to echelon form

> aug1 := array(1..3,1..3,[[1,3,2],[2,4,6],[3,2,6]]);

\[
\text{aug1} := \begin{bmatrix}
-1 & 3 & 2 \\
2 & 4 & 6 \\
3 & 2 & 6
\end{bmatrix}
\]

> gausselim(aug1);
\[
\begin{bmatrix}
-1 & 3 & 2 \\
0 & 10 & 10 \\
0 & 0 & 1
\end{bmatrix}
\]
No solution possible, since last row is all zeros except for last column. xvec is not in span!

6.2 Problem 11b)
> solve(\{-c1+3*c2=-9,2*c1+4*c2=-2,3*c1+2*c2=5\},\{c1,c2\});
\{c2 = -2, c1 = 3\}

> aug2 := array(1..3,1..3,[[1,3,-9],[2,4,-2],[3,2,5]]);

\[
\text{aug2} := \begin{bmatrix}
-1 & 3 & -9 \\
2 & 4 & -2 \\
3 & 2 & 5
\end{bmatrix}
\]

> gausselim(aug2);
\[
\begin{bmatrix}
-1 & 3 & -9 \\
0 & 10 & -20 \\
0 & 0 & 0
\end{bmatrix}
\]
7 Problem 14

7.1 Problem 14a)

\[
\text{mat} := \begin{bmatrix}
1 & 0 & -2 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\]

\[\text{mat} \text{ is singular, so vectors are not a basis. Also, it is easy to see that anything with an "x" term cannot be created as a linear combination of the vectors.}\]

7.2 Problem 14b)

\[
\text{mat} := \begin{bmatrix}
2 & 0 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[\text{gausselim(mat)};\]

\[
\begin{bmatrix}
2 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

The echelon form of mat is such that we can always find a solution (infinitely many actually) to the augmented system.

7.3 Problem 14c)

\[
\text{mat} := \begin{bmatrix}
2 & 1 & -1 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[\text{gausselim(mat)};\]

\[
\begin{bmatrix}
2 & 1 & -1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The echelon form of mat is such that we can always find a solution (infinitely many actually) to the augmented system.
7.4 Problem 14d)

\[ \text{mat} := \text{array}(1..3,1..2,[[2,-1],[1,0],[0,1]]); \]
\[
\begin{bmatrix}
2 & -1 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[ \text{gausselim(mat);} \]
\[
\begin{bmatrix}
2 & -1 \\
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

The echelon form of mat is such that we can never find a solution to the augmented system.
Therefore the vectors do not span.