Part 1. Multiple choice problems. Each problem is worth 4 points. Read each question carefully. No calculators are allowed on this part.

1. Let \( f(x) = e^{7x} \). Find \( f'(x) \).
   
a) \( 7e^{7x} \)  
b) \( e^{7x} \)  
c) \( \frac{e^{7x}}{7} \)  
d) \( 7xe^{7x-1} \)  
e) \( 7xe^{6x} \)

   \[
   \frac{d}{dx} e^{7x} = e^{7x} \frac{d}{dx} (7x) = 7e^{7x}
   \]

2. What is the domain of the function \( f(x) = \ln(e^x - 2) \)?
   
a) \( (-\infty, \infty) \)  
b) \( (0, \infty) \)  
c) \( [0, \infty) \)  
d) \( (\ln 2, \infty) \)  
e) \( [\ln 2, \infty) \)

   The domain of \( \ln u \) is \( u > 0 \). Thus, the domain of \( \ln(e^x - 2) \) is the set of \( x \) for which \( e^x - 2 > 0 \)

   \[
   e^x - 2 > 0 \implies e^x > 2 \implies x > \ln 2 \implies (\ln 2, \infty)
   \]

3. Let \( f(x) = x^5 + 6x^2 + 10 \). Find \( f''(0) \).
   
a) \(-12\)  
b) \(6\)  
c) \(0\)  
d) \(12\)  
e) \(1\)

   \[
   f''(0) = \frac{d^2}{dx^2} (x^5 + 6x^2 + 10) \bigg|_{x=0}
   \]

   \[
   = \frac{d}{dx} (5x^4 + 12x) \bigg|_{x=0} = (20x^3 + 12) \bigg|_{x=0}
   \]

   \[
   = 12
   \]
4. \( \frac{d}{dx} \left( \frac{\cos x}{x} \right) = \)
   a) \( -\frac{\sin x}{x^2} \)  
   b) \( -\frac{x \sin x - \cos x}{x^2} \)  
   c) \( \frac{x \sin x + \cos x}{x^2} \)  
   d) \( \frac{x \sin x - \cos x}{x^2} \)  
   e) \( -\frac{\sin x}{1} \)

   \[ \frac{d}{dx} \left( \frac{\cos x}{x} \right) = -\frac{x \sin x - \cos x}{x^2} \]

5. Compute \( \lim_{x \to \infty} \frac{2^x + 2^{-3x}}{5(2^x) - 2^{-3x}} \)

   a) 1  
   b) \(-1\)  
   c) 4  
   d) \(\frac{1}{2}\)  
   e) none of these

   \[ \lim_{x \to \infty} \frac{2^x + 2^{-3x}}{5(2^x) - 2^{-3x}} = \lim_{x \to \infty} \frac{1 + 2^{-4x}}{5 - 2^{-4x}} = \frac{1}{5} \]

   Thus, the correct answer is e).

6. Let \( f(x) = (2x^2 - 7x)^4 \). Compute the derivative of \( f \).

   a) \( (4x - 7)^4 \)  
   b) \( 16(4x - 7)^3 \)  
   c) \( 4(2x^2 - 7x)^3 \)  
   d) \( (16x - 28)(2x^2 - 7x)^3 \)  
   e) \( 16x(2x^2 - 7x)^3 \)

   \[ \frac{d}{dx} (2x^2 - 7x)^4 = 4(2x^2 - 7x)^3 \frac{d}{dx}(2x^2 - 7x) = 4(2x^2 - 7x)^3 (4x - 7) \]

7. \( \lim_{x \to 0} \frac{\sin x + \cos x - 1}{2x} \)

   a) 0  
   b) 1  
   c) \( \frac{1}{2} \)  
   d) \( \frac{3}{4} \)  
   e) does not exist

   \[ \lim_{x \to 0} \frac{\sin x + \cos x - 1}{2x} = \lim_{x \to 0} \frac{\sin x}{2x} + \lim_{x \to 0} \frac{\cos x - 1}{2x} = \frac{1}{2} + 0 = \frac{1}{2} \]
8. Find the linear (i.e., tangent line) approximation to the function \( f(x) = (1 + x)^{1/3} \) at the point \( x = 0 \).

a) \( 1 + x^{1/3} \)  

b) \( 1 + \frac{x}{3} \)  

c) \( 1 + x \)  

d) \( \frac{x}{3} \)  

e) \( 1 - \frac{2x}{9} \)

\[ y = f(0) + f'(0)(x - 0) = 1 + \frac{1}{3}x \]

9. The graph of a differentiable function \( f \) passes through the point \( P(2, 1) \). If one begins Newton’s method (for finding the roots of \( f(x) = 0 \)) with \( x_1 = 2 \), then one obtains \( x_2 = 7/4 \). What is the slope of the tangent line to the graph of \( f \) at the point \( P \)?

a) 2  

b) \(-4\)  

c) 4  

d) \(\frac{7}{4}\)  

e) insufficient information given

The problem asks for the value of the derivative of \( f \) at \( x = 2 \). Since Newton’s method was used to get \( x_2 = 7/4 \), we have

\[ \frac{7}{4} = x_2 - x_1 = \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{f'(2)}. \]

Thus, \( f'(2) = 4 \)

10. Find the inverse of the function \( y = e^{x^3} \).

a) \( \ln (x^{1/3}) \)  

b) \( (\ln x)^{1/3} \)  

c) \( \ln (x^3) \)  

d) \( (\ln x)^3 \)  

e) \( e^{-x^3} \)

\[ y = e^{(x^3)} \implies \ln y = x^3 \implies x = (\ln y)^{1/3} \]

Interchanging the roles of \( x \) and \( y \) \( \implies y = (\ln x)^{1/3} \)

11. Find the slope of the tangent line to the curve \( x^3y^2 + 2y = 3 \) at the point \( (1, 1) \).

a) 0  

b) \(-\frac{3}{4}\)  

c) \(\frac{3}{8}\)  

d) \(\frac{3}{5}\)  

e) \(-\frac{5}{2}\)

Differentiating the equation implicitly we have,

\[ 3x^2y^2 + 2x^3yy' + 2y' = 0 \]

\[ y'|_{(1,1)} = \frac{-3x^3y^2}{2x^3y + 2}|_{(1,1)} = \frac{-3}{4} \]
12. A spherical balloon is inflated with helium at a rate of $100\pi \text{ ft}^3/\text{min}$. How fast is the radius increasing at the instant the radius is 5 ft. The volume of a sphere of radius $r$ is $V = \frac{4}{3}\pi r^3$.

a) 1 ft/min  b) 10 ft/min  c) 20 ft/min  d) 40$\pi$ ft/min  e) 10$\pi$ ft/min

Differentiating the volume/radius equation with respect to $t$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} \bigg|_{r=5} = \frac{dV/dt}{4\pi r^2} \bigg|_{r=5} = \frac{100\pi}{4\pi \times 25} = 1$$

Part 2. Worked out problems. Show all work for full credit. You may not use your calculator on this part of the examination until all Scantron forms are collected. The point value of each problem is shown just before the statement of the problem.

13. (9) Let $g(x) = f(x^2 - \sqrt{x})$. A table of values of $f(x)$ and $f'(x)$ for various values of $x$ is given below. Use this table to find $g'(4)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-2</td>
<td>3</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

$$g'(x) = f'(x^2 - \sqrt{x}) \frac{d}{dx}(x^2 - \sqrt{x}) = f'(x^2 - \sqrt{x}) \left(2x - \frac{1}{2}x^{-1/2}\right)$$

$$g'(4) = f'(14)(8 - \frac{1}{4}) = 5 \frac{31}{4} = \frac{155}{4}$$
14. (9) Suppose $C$ is the parametric curve determined by the vector equation
\[ \vec{r}(t) = \langle \sin 2t, \cos^2 t \rangle \quad 0 \leq t \leq \frac{\pi}{2}. \]

(a) Find $\vec{r}''(t)$, $0 < t < \frac{\pi}{2}$.

\[ \vec{r}'(t) = \langle 2 \cos 2t, -2 \sin t \cos t \rangle \]

(b) Find the unique number $t_0$ in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\vec{r}(t_0) = \langle 1, 1/2 \rangle$.

From the equation $\sin 2t = 1$, we have $2t = \pi/2$, or $t = \pi/4$. At $t = \pi/4$ we also have $\cos^2(\pi/4) = 1/2$. Thus, the number we seek is $\pi/4$.

(c) Determine the cosine of the angle between the position vector of the point $P(1, 1/2)$ and a tangent vector to the curve $C$ at the point $P$.

If $\theta$ denotes the angle, then we have
\[
\cos \theta = \frac{\vec{r}'(\pi/4) \cdot \vec{r}(\pi/4)}{||\vec{r}'(\pi/4)|| \cdot ||\vec{r}(\pi/4)||} = \frac{\langle 0, -1 \rangle \cdot \langle 1, 1/2 \rangle}{||\langle 0, -1 \rangle|| \cdot ||\langle 1, 1/2 \rangle||}
\]
\[
= \frac{-1/2}{\sqrt{5}/4} = \frac{-1}{\sqrt{5}}
\]
Consider the parametric curve given by the following pair of equations:

\[ x(t) = t^5 + 2t, \quad y(t) = 2t^3 - 3t^2 + 4, \quad -\infty < t < \infty. \]

(a) Find \( \frac{dy}{dx} \) in terms of \( t \).

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2 - 6t}{5t^4 + 2}
\]

(b) Find all values of the parameter \( t \) for which the curve has a horizontal tangent.

The curve has a horizontal tangent if \( dy/dx = 0 \). Thus, we need \( dy/dt = 0 \).

\[
6t^2 - 6t = 0 \iff 6t(t - 1) = 0 \iff t = 0 \text{ or } t = 1
\]

(c) What is the Cartesian equation for each of the horizontal tangents from part b. above?

We need the \( y \) values which correspond to \( t = 0 \) and \( t = 1 \). They are

\[
y(0) = 4 \quad \text{and} \quad y(1) = 3
\]

Thus, the equations for the two horizontal tangents are \( y = 3 \) and \( y = 4 \).
16. (9) Two ships leave the same port at noon. Ship A sails north at 20 mph and ship B sails east at 15 mph. How fast is the distance between them changing at 2:00 p.m?

At 2:00 p.m. ship A is 40 miles north, and ship B is 30 miles east of the port. So at 2:00 p.m. the two ships are \(\sqrt{40^2 + 30^2} = 50\) miles apart. If \(D\) represents the distance between them; \(x\) the distance ship B is from port, and \(y\) the distance ship A is from port, then

\[
D^2 = x^2 + y^2
\]

\[
2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]

Thus,

\[
\left. \frac{dD}{dt} \right|_{t=2} = \frac{30}{50} \cdot 15 + \frac{40}{50} \cdot 20 = 25 \text{ miles per hour}.
\]

17. (8) Find the values of the constant \(r\) for which \(y = e^{rx}\) satisfies

\[
y + \frac{dy}{dx} = \frac{d^2y}{dx^2}.
\]

Substituting \(e^{rx}\) for \(y\) in the differential equation we get

\[
e^{rx} + re^{rx} = r^2 e^{rx}
\]

\[
1 + r = r^2
\]

The solutions to this quadratic equation are

\[
r = \frac{1 \pm \sqrt{5}}{2}.
\]

18. (8) Find all values of \(x\) for which \(e^{\sqrt{\ln x}} = x^{1/10}\).

\[
e^{\sqrt{\ln x}} = x^{1/10} \implies e^{\sqrt{\ln x}} = e^{\ln x/10} \implies
\]

\[
\sqrt{\ln x} = \frac{\ln x}{10} \implies \ln x = 0 \text{ or } \ln x = 100 \implies
\]

\[
x = 1 \text{ or } x = e^{100}.
\]