1. Consider the line whose vector equation is \( \vec{r}(t) = (2 - 3t)\mathbf{i} + (-1 - t)\mathbf{j} \)
Which vector is parallel to this line?

(a) \( (2, -1) \)  \hspace{1cm} (b) \( (-3, -1) \)  \hspace{1cm} (c) \( \mathbf{j} \)  \hspace{1cm} (d) \( (-1 - t)\mathbf{j} \)  \hspace{1cm} (e) \( \mathbf{i} \)

The easiest way to do this problem is to rewrite the expression for \( \vec{r}(t) \).

\[
\vec{r}(t) = (2, -1) + t(-3, -1)
\]

Thus, \((-3, -1)\) is parallel to the line.

2. Find a unit vector which points in the same direction as \( (-1, 5) \).

(a) \( (1, -5) \)  \hspace{1cm} (b) \( (-1, 5) \)  \hspace{1cm} (c) \( \frac{-1, 5}{26} \)  \hspace{1cm} (d) \( \frac{-1, 5}{\sqrt{26}} \)  \hspace{1cm} (e) \( \frac{1, -5}{\sqrt{26}} \)

To find a unit vector which points in the same direction as a given vector, divide the vector by its length.

\[
\text{unit vector} = \frac{\langle -1, 5 \rangle}{\sqrt{1 + 5^2}} = \frac{\langle -1, 5 \rangle}{\sqrt{26}}
\]

3. Which of the following is true about the lines

\( \vec{r}_1(t) = (1 + 3t)\mathbf{i} + (2 + 4t)\mathbf{j} \)  \hspace{1cm} \( \vec{r}_2(t) = (1 - 4t)\mathbf{i} + (2 + 3t)\mathbf{j} \)

(a) They are parallel.
(b) They intersect at the point \((-3, 4)\).
(c) They intersect at the point \((3, 4)\).
(d) They intersect at the point \((1, -2)\).
(e) They intersect at the point \((1, 2)\).

Rewrite the equations of the lines

\[
\vec{r}_1(t) = (1, 2) + t(3, 4) \\
\vec{r}_2(t) = (1, 2) + t(-4, 3)
\]

Thus, the lines intersect at the point \((1, 2)\).
4. What is the magnitude of the projection of the vector \( \langle 4, 3 \rangle \) onto the vector \( \langle 1, 3 \rangle \)?

(a) \( \frac{13}{\sqrt{10}} \)  (b) \( \frac{11}{\sqrt{10}} \)  (c) \( \frac{13}{25} \)  (d) \( \frac{11}{10} \)  (e) \( \frac{13}{5} \)

The projection of \( \langle 4, 3 \rangle \) onto \( \langle 1, 3 \rangle \) equals

\[
\text{Proj}_{\langle 1, 3 \rangle} \langle 4, 3 \rangle = \frac{\langle 1, 3 \rangle \cdot \langle 4, 3 \rangle}{\| \langle 1, 3 \rangle \| \| \langle 1, 3 \rangle \|} = \frac{13}{\sqrt{10} \| \langle 1, 3 \rangle \|}
\]

Since the vector \( \langle 1, 3 \rangle \) has length 1, the magnitude of the projection equals \( 13/\sqrt{10} \).

5. Suppose that \( h(x) = f(x)g(x) - 5x \) and \( f(4) = 3, f'(4) = -1, g(4) = 2, \) and \( g'(4) = 5 \). What is \( h'(4) \)?

(a) 13  (b) -5  (c) 8  (d) -10  (e) -7

\[
h'(x) = f'(x)g(x) + f(x)g'(x) - 5
\]
\[
h'(4) = f'(4)g(4) + f(4)g'(4) - 5
\]
\[
= (-1)(2) + (3)(5) - 5
\]
\[
= 8
\]
6. Find the equation of the tangent line to the curve $y = 5x^2 + 7$ at $(2, 27)$.

(a) $y - 27 = (x - 2)$  
(b) $y - 2 = 20(x - 2)$  
(c) $y - 27 = 10x(x - 2)$

(d) $y - 27 = 20(x - 2)$  
(e) $y - 27 = 10(x - 2)$

The equation of the tangent line has the form $y - 27 = m(x - 2)$, where $m$ is the slope of the tangent line. $m$ is also equal to the value of the derivative of $y$ at $x = 2$.

$$m = \left. \frac{dy}{dx} \right|_{x=2} = 10x|_{x=2} = 20$$

Thus, the equation of the tangent line is $y - 27 = 20(x - 2)$.

7. Determine $\lim_{x \to 3} \frac{x^2 + 4x - 21}{x - 3}$

(a) 0  
(b) $\infty$  
(c) 1  
(d) 10  
(e) Does not exist.

$$\lim_{x \to 3} \frac{x^2 + 4x - 21}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 7)}{x - 3} = \lim_{x \to 3} (x + 7) = 10$$

8. If $f(x) = x^3 - \sqrt{x}$, what is $f'(x)$?

(a) $3x^3 - \sqrt{x}$  
(b) $3x^2$  
(c) $3x^3 - \frac{1}{2}x^{-1/2}$  
(d) $3x^2 - \frac{1}{2}x^{-1/2}$  
(e) $3x^2 + \frac{1}{2}x^{-1/2}$

$$f'(x) = 3x^2 - \frac{1}{2}x^{-1/2}$$
9. Find the values of \( f(x) \) where \( f(x) \) is discontinuous

\[
f(x) = \begin{cases} 
  x + 2 & \text{if } x < -3 \\
  x^2 - 10 & \text{if } -3 \leq x \leq 3 \\
  1 - 2x & \text{if } 3 < x 
\end{cases}
\]

(a) -3 only   (b) 0 only   (c) 3 only   (d) -3 and 3   (e) 3 and 0 and -3

There are only two values of \( x \) ( -3 and 3 ) where this function can be discontinuous. we will check the one-sided limits to determine if the function is continuous at these points.

\[
\begin{align*}
\lim_{x \to -3^-} f(x) & = \lim_{x \to -3^-} x + 2 = -1 \\
\lim_{x \to -3^+} f(x) & = \lim_{x \to -3^+} x^2 - 10 = -1 = f(-3) \\
\lim_{x \to 3^-} f(x) & = \lim_{x \to 3^-} x^2 - 10 = -1 = f(3) \\
\lim_{x \to 3^+} f(x) & = \lim_{x \to 3^+} 1 - 2x = -5
\end{align*}
\]

Thus, \( f(x) \) is not continuous at \( x = 3 \), and is continuous at \( x = -3 \).

10. Suppose that \( f(x) = x + \frac{1}{x} \). What does \( \frac{f(3 + h) - f(3)}{h} \) equal?

(a) \( 1 - \frac{1}{3(3 + h)} \)   (b) \( \frac{(x + 1/x)(3 + h) - (3 + 1/3)}{h} \)

(c) \( \frac{(x + 1/x)(3 + h) - (x + 1/x)(3 + 1/3)}{h} \)   (d) \( \frac{h - 1/3}{h} \)   (e) 1

\[
\frac{f(3 + h) - f(3)}{h} = \frac{(3 + h + \frac{1}{3 + h}) - (3 + \frac{1}{3})}{h} = \frac{h + \frac{1}{3 + h} - \frac{1}{3}}{h} = \frac{h + \frac{-h}{h(3 + h)}}{h(3 + h)} = 1 - \frac{1}{h(3 + h)}
\]
11. Evaluate \( \lim_{x \to \infty} \left( \frac{3x^4 + 6x - 8}{17x + 8x^4} \right) \)

(a) \( \frac{3}{4} \)  
(b) \( \frac{3}{8} \)  
(c) \(-1\)  
(d) \( \frac{3}{17} \)  
(e) \( \frac{6}{17} \)

\[
\lim_{x \to \infty} \left( \frac{3x^4 + 6x - 8}{17x + 8x^4} \right) = \lim_{x \to \infty} \frac{x^4 (3 + 6/x^3 - 6/x^4)}{x^4 (17/x^3 + 8)} \\
= \lim_{x \to \infty} \frac{(3 + 6/x^3 - 6/x^4)}{(17/x^3 + 8)} \\
= \frac{3}{8}
\]

12. Which of the following statements regarding the function \( f(x) \) whose graph is shown below is TRUE?

(a) \( f \) has a limit at both \( x = -3 \) and \( x = 2 \).
(b) \( f \) is differentiable at \( x = 0 \) and at \( x = 2 \).
(c) \( f \) has a limit at \( x = -3 \) and is differentiable at \( x = 0 \).
(d) \( f \) has a limit at \( x = 2 \) and is not differentiable at \( x = 0 \).
(e) \( f \) has a limit at \( x = 2 \) and is differentiable at \( x = 0 \).

The only one of these five items which is true is number (e).
Part 2. Worked out problems. Show all work for full credit. You may not use your calculator on this part of the examination until all Scantron forms are collected. Each problem is worth 7 points.

13. (a) Give the definition in terms of limits for the derivative of \( f(x) \) at the point \( a \).

Either of the following is acceptable

\[
\begin{align*}
\frac{df}{dx} &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
\frac{df}{dx} &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\end{align*}
\]

(b) Find all values of \( c \) such that the function

\[
g(x) = \begin{cases} 
  x^2 + cx & \text{if } x < 2 \\
  6cx - 14 & \text{if } x \geq 2
\end{cases}
\]

is continuous at \( x = 2 \).

The left and right hand limits as \( x \) approaches 2 of \( g(x) \) must be equal. Thus,

\[
\begin{align*}
\lim_{x \to 2^+} g(x) &= 12c - 14, \quad \lim_{x \to 2^-} g(x) = 4 + 2c^2
\end{align*}
\]

Set these two expressions equal to each other and solve for \( c \).

\[
\begin{align*}
12c - 14 &= 4 + 2c^2 \\
c^2 - 6c + 9 &= 0 \\
(c-3)^2 &= 0
\end{align*}
\]

Thus, \( c = 3 \) is the only value of \( c \) for which the function \( g(x) \) is continuous at \( x = 2 \).

(c) For the determined values of \( c \), is the function differentiable at \( x = 2 \)?

For \( c = 3 \)

\[
g(x) = \begin{cases} 
  x^2 + 9x & \text{if } x < 2 \\
  18x - 14 & \text{if } x \geq 2
\end{cases}
\]

The derivative of \( g(x) \) exists for all \( x \neq 2 \). At \( x = 2 \), the left and right hand difference quotients are

\[
\begin{align*}
\lim_{x \to 2^+} \frac{g(x) - g(2)}{x - 2} &= \lim_{x \to 2^+} \frac{(18x - 14) - (22)}{x - 2} = 18 \\
\lim_{x \to 2^-} \frac{g(x) - g(2)}{x - 2} &= \lim_{x \to 2^-} \frac{(x^2 + 9x) - (22)}{x - 2} = 13
\end{align*}
\]
Since the right and left hand derivatives are not equal, the function is not differentiable at \( x = 2 \).

A second way to work this part of the problem is to calculate \( g'(x) \)

\[
g'(x) = \begin{cases} 
2x + 9 & \text{if } x < 2 \\
18 & \text{if } x > 2 
\end{cases}
\]

and observe that when \( x = 2 \) the two different branches of \( g'(x) \) are not equal. Thus, \( g' \) does not exist at \( x = 2 \).

14. Using the definition of the derivative, find \( f'(x) \) for \( f(x) = \frac{5}{x^2} \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
= \lim_{h \to 0} \frac{1}{h} \left( \frac{5}{(x+h)^2} - \frac{5}{x^2} \right) \\
= \lim_{h \to 0} \frac{1}{h} \left( \frac{5x^2 - 5(x^2 + 2xh + h^2)}{x^2(x+h)^2} \right) \\
= \lim_{h \to 0} \frac{-10xh - 5h^2}{hx^2(x+h)^2} \\
= \lim_{h \to 0} \frac{-10x - 5h}{hx^2} \\
= \frac{-10x}{x^4} \\
= -\frac{10}{x^3}
\]

15. Given that \( f(4) = -1 \) and \( f'(4) = 3 \), find the derivative of \( g(x) = \frac{f(x)}{x} \) at \( x = 4 \).

\[
g'(x) = \frac{f'(x)x - f(x)}{x^2} \\
g'(4) = \frac{f'(4)4 - f(4)}{16} \\
= \frac{3 \cdot 4 - (-1)}{16} \\
= \frac{13}{16}
\]
16. Find the points where the instantaneous rate of change of \( f(x) = 3x - \frac{x^2 - 9}{x} \) equals 1.

Remember: the instantaneous rate of change of a function is found by taking the derivative of the function. Before taking the derivative of \( f(x) \) simplify it. \( f(x) = 3x - x + 9/x = 2x + 9/x. \)

\[
f'(x) = 2 - \frac{9}{x^2} = 1
\]

\[
1 = \frac{9}{x^2}
\]

\[
x^2 = 9
\]

\[
x = \pm 3
\]

17. (a) Two dragonflies pull on an object with forces of 30 dynes and 60 dynes respectively. The forces \( F_1 \) and \( F_2 \) make angles of 30\(^\circ\) and 150\(^\circ\) degrees respectively with the positive \( x \)-axis. Find the magnitude and direction of the resultant force, \( F = F_1 + F_2. \)

\[
F_1 = 30 (\cos 30, \sin 30) \approx (25.9807, 15)
\]

\[
F_2 = 60 (\cos 150, \sin 150) \approx (-51.962, 30.0)
\]

\[
F = F_1 + F_2 = (25.9807, 15) + (-51.962, 30.0)
\]

\[
= (-25.981, 45.0)
\]

The magnitude of \(|F| \approx |(-25.981, 45.0)| \approx 51.962, and the direction is approximately \( 90 + \frac{180}{\pi} \arctan \left( \frac{25.981}{45} \right) \approx 90 + 30.00022716 \approx 120.0002 \)
If you do exact arithmetic, the solution is

\[
\mathbf{F} = (30 \cos 30 + 60 \cos 150, 30 \sin 30 + 60 \sin 150) \\
= 60 \left( \frac{\cos 30}{2} + \cos 150, \frac{\sin 30}{2} + \sin 150 \right) \\
= 60 \left( \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} + \frac{1}{4}, \frac{1}{4} + \frac{1}{2} \right) \\
= 60 \left( -\frac{\sqrt{3}}{4}, \frac{3}{4} \right) \\
= 15\sqrt{12} \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)
\]

Thus, the magnitude is \(15\sqrt{12}\), and the force makes an angle of 120 degrees with the positive \(x\) axis.
(b) Sketch the resultant force on the diagram below.
18. Use the graphical information below about the functions $f(x)$ and $g(x)$ to determine $\frac{d}{dx} \left[ \frac{f}{g} \right]_{x=1}$.

From the graph we see that $f(1) = 1$, $f'(1) = \frac{4}{2} = 2$, $g(1) = 2$, and $g'(1) = 0$.

$$\frac{d}{dx} \left[ \frac{f}{g} \right]_{x=1} = \frac{f'(1)g(1) - f(1)g'(1)}{g^2(1)} = \frac{2 \cdot 2 - 0}{4} = 1$$

A second way to work this problem is to note that $g(x)$ is constant in an interval containing 1. Thus, the ratio $f/g$ becomes $f/2$ and the derivative just equals $f'(1)/2 = 1$. 
Bonus Problem:

\[ f(x) \] is continuous on the interval \([0, 1]\), and for every \(x\) in this interval we have \(0 \leq f(x) \leq 1\). Let \(x_0 = \frac{1}{2}\). Define the following points

\[
\begin{align*}
x_1 &= f(x_0) \\
x_2 &= f(x_2) \\
&\quad \vdots \\
x_{n+1} &= f(x_n)
\end{align*}
\]

Suppose that \(\lim_{n \to \infty} x_n\) exists and equals a number which we denote by \(x_\infty\). That is, \(\lim_{n \to \infty} x_n = x_\infty\). Show that \(x_\infty = f(x_\infty)\).

The key to understanding this is continuity. If a sequence of numbers \(x_n\) gets close to a number \(x_\infty\), since \(f(x)\) is continuous, it must be true that \(f(x_n)\) will get close to \(f(x_\infty)\). Thus, from

\[ x_{n+1} = f(x_n) \]

we see that the left hand side converges to \(x_\infty\) and the right hand side converges to \(f(x_\infty)\). This implies that

\[ x_\infty = f(x_\infty) \]