Part 1. Multiple choice problems. Each problem is worth 4 points. Read each question carefully. No calculators are allowed on this part.

1. Find \( \lim_{x \to 0} \frac{x}{\sin (3x)} \).
   A) 3  B) \(-\frac{1}{\cos (3x)}\)  C) \(\frac{1}{3 \cos (3x)}\)  D) \(\frac{1}{3}\)  E) undefined
   \[
   \lim_{x \to 0} \frac{x}{\sin (3x)} = \frac{1}{3} \lim_{x \to 0} \frac{3x}{\sin (3x)} = \frac{1}{3}
   \]

2. Let \( f(x) = \frac{1}{x} \). What is the linear approximation to \( f(x) \) at \( x = 2 \)?
   A) \(\frac{1}{2} - \frac{1}{4}(x - 2)\)  B) \(\frac{1}{2} + \frac{1}{4}(x - 2)\)  C) \(\frac{1}{2} + \frac{1}{x - 2}\)
   D) \(\frac{1}{2} - \frac{1}{4} \frac{1}{x - 2}\)  E) \(\frac{2}{1 + x}\)
   \[
   f(x) \approx f(2) + f'(2)(x - 2) = \frac{1}{2} - \frac{1}{4}(x - 2)
   \]

3. Find \( \frac{d}{dx} [(9 - x^3)^{10}] \) at \( x = 2 \).
   A) 10  B) -120  C) \(9^{10}(-3x^2)\)  D) -30  E) \((-3x^2)^{10}\)
   \[
   \frac{d}{dx} [(9 - x^3)^{10}] \bigg|_{x=2} = 10 \left(9 - x^3\right)^9 \left(-3x^2\right) \bigg|_{x=2} = 10(-12) = -120
   \]

4. Find the derivative of \( \sin (5x) \).
   A) \(\cos 5x\)  B) \(-\cos 5x\)  C) \(5 \cos 5x\)  D) \(-5 \cos 5x\)  E) \(-5 \sin 5x\)
   \[
   \frac{d}{dx} [\sin (5x)] = 5 \cos (5x)
   \]
5. Evaluate \( \lim_{x \to 0} \frac{1 - \cos x}{x} \). 

A) 1  B) 0  C) \( \frac{1}{2} \)  D) \(-1\)  E) undefined

\[
\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \left[ \frac{1 + \cos x}{1 + \cos x} \right] \\
= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} \\
= \left[ \lim_{x \to 0} \left( \frac{\sin x}{x} \right) \right] \left[ \lim_{x \to 0} \left( \frac{\sin x}{(1 + \cos x)} \right) \right] \\
= 1 \times 0 = 0
\]

6. What is the second derivative of the function \( f(x) = (x^2 + 1) \sin x \)?

A) \(2 \cos x + 2x \cos x\)  B) \(2 \sin x + 2x \cos x\)  C) \(2x \cos x\)

D) \(2 \cos x - 2x \sin x\)  E) \(2 \sin x + 4x \cos x - (x^2 + 1) \sin x\)

\[
\frac{d^2}{dx^2} \left[ (x^2 + 1) \sin x \right] = \frac{d}{dx} \left[ 2x \sin x + (x^2 + 1) \cos x \right] \\
= 2 \sin x + 2x \cos x + 2x \cos x - (x^2 + 1) \sin x \\
= 2 \sin x + 4x \cos x - (x^2 + 1) \sin x
\]

7. Calculate the 19th derivative of \( \sin 3x \).

A) \(\sin 3x\)  B) \(3^{19} \cos 3x\)  C) \(-(3^{19}) \cos 3x\)

D) \(-(3^{19}) \sin 3x\)  E) \(3^{18} \cos 3x\)

The answer must have a \(3^{19}\) in it as each differentiation increases the number of 3’s by 1. So our only problem is to keep track of whether we wind up with a \(\pm \sin 3x\) or a \(\pm \cos 3x\). It is also clear that every 4th derivative cycles back to \(\sin 3x\). Thus, we have

\[
\frac{d^{16}}{dx^{16}} (\sin 3x) = 3^{16} \sin 3x
\]

Now take three more derivative. This tranforms \(\sin 3x\) into \(- \cos 3x\). Thus

\[
\frac{d^{19}}{dx^{19}} (\sin 3x) = -(3^{19}) \cos 3x
\]
8. Let a curve be parametrized by $\mathbf{R}(t) = (t, t^3 + 3)$. Which of the following is a unit tangent vector at $t = 1$?

A) $\langle 1, 3 \rangle$  B) $\langle 1, 2 \rangle$  C) $\frac{\langle 1, 3 \rangle}{\sqrt{10}}$  D) $\frac{\langle 1, 1 \rangle}{\sqrt{2}}$  E) $\frac{\langle 1, 6 \rangle}{\sqrt{37}}$

$$\frac{d}{dt} \mathbf{R}(t) \bigg|_{t=1} = \frac{d}{dt} (t, t^3 + 3) \bigg|_{t=1} = \langle 1, 3t^2 \rangle \bigg|_{t=1} = \langle 1, 3 \rangle$$

Thus, a unit vector in this direction is

$$\frac{\langle 1, 3 \rangle}{\sqrt{10}}$$

9. Find the derivative of the function $f(x) = e^{5x}$.

A) $5e^{5x}$  B) $e^{5x}$  C) $\frac{e^{5x}}{5}$  D) $5xe^{5x-1}$  E) $5xe^{4x}$

$$\frac{d}{dx} e^{5x} = 5e^{5x}$$

10. Evaluate $\lim_{x \to \infty} \frac{3^x + 3^{-2x}}{2(3^x) - 3^{-2x}}$.

A) $\frac{1}{2}$  B) $1$  C) $-1$  D) $\frac{1}{3}$  E) undefined

$$\lim_{x \to \infty} \frac{3^x + 3^{-2x}}{2(3^x) - 3^{-2x}} = \lim_{x \to \infty} \frac{3^x (1 + 3^{-5x})}{2(3^x) (2 - 3^{-5x})} = \lim_{x \to \infty} \frac{(1 + 3^{-5x})}{(2 - 3^{-5x})} = \frac{1}{2}$$
11. Suppose \( f(x) \) is one-to-one, and has an inverse function \( g(x) \). A plot of \( f(x) \) is shown below. From this plot determine \( g(4) \).

\[
\begin{array}{c|c|c|c|c|c}
\text{A)} & \text{B)} & \text{C)} & \text{D)} & \text{E)} \\
\text{2} & \text{2.5} & \text{4} & \text{5} & \text{Not enough information given to answer question.} \\
\end{array}
\]

Since \( f(5) = 4 \), \( g(4) = 5 \).

12. Find \( \frac{d}{dx} \tan x \)

A) \( \sin x \) \hspace{1cm} \text{B)} \( \sec x \) \hspace{1cm} \text{C)} \( -\cot x \) \hspace{1cm} \text{D)} \( \sec^2 x \) \hspace{1cm} \text{E)} \( \csc^2 x \)

\[
\frac{d}{dx} \tan x = \sec^2 x
\]

Part 2. Worked out problems. Show all work for full credit. You may not use your calculator on this part of the examination until all Scantron forms are collected. Each problem is worth 7 points.

13. Find the equation of the tangent line to the curve \( x^4 + xy + y^4 = 19 \) at \( (2, 1) \).

Differentiate the equation, assuming that \( y \) is a function of \( x \).

\[
4x^3 + y + xy' + 4y^3 y' = 0
\]

\[
y'(2,1) = \left. \frac{-y - 4x^3}{x + 4y^3} \right|_{(2,1)} = \frac{-1 - 32}{2 + 4} = \frac{-33}{6} = -\frac{11}{2}
\]

Thus, an equation for the tangent line is

\[
y - 1 = -\frac{11}{2} (x - 2)
\]
14. The position of a weight attached to a vertical spring is given by \( y(t) = 10 \sin(3t) \), for \( 0 \leq t \leq \frac{\pi}{3} \). \( y \) has units of feet and \( t \) has units of seconds.

(a) Find the velocity and acceleration as functions of \( t \).

\[
\text{velocity} = \frac{dy}{dt} = 30 \cos(3t) \text{ ft/sec}
\]
\[
\text{acceleration} = \frac{d^2y}{dt^2} = -90 \sin(3t) \text{ ft/sec}^2
\]

(b) What is the acceleration at the time \( t \) when the speed is a minimum?

The smallest possible value of the speed 0 and this occurs when \( \cos(3t) = 0 \). That is, when
\( 3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \), or when \( t = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \ldots \). The only one of these values which is in the interval \([0, \pi/3]\) is \( \pi/6 \). The acceleration at this moment equals
\[
-90 \sin \left( 3 \left( \frac{\pi}{6} \right) \right) = -90 \sin \left( \frac{\pi}{2} \right) = -90 \text{ ft/sec}^2
\]

15. Given the parametric curve \( x(t) = 2t + t^2, y(t) = t^3 + t \), find the points on the curve, if any, where the tangent to the curve makes an angle of 45° with the positive \( x \)-axis.

The tangent line makes an angle of 45 degrees with the positive \( x \)-axis when \( x = y \). That is,
\[
2 + 2t = 3t^2 + 1
\]
\[
3t^2 - 2t - 1 = 0
\]
\[
t = \frac{2 \pm \sqrt{4 + 12}}{6} = \frac{1 \pm \sqrt{3}}{3}
\]
\begin{align*}
&= 1 \quad \text{or} \quad -\frac{1}{3}
\end{align*}

The points corresponding to these values of \( t \) are
\[
t = 1: \quad (3, 2), \quad t = -\frac{1}{3}: \quad \left( \frac{5}{9}, \frac{-10}{27} \right)
\]

16. A 4 meter ladder leans against a wall. If the foot of the ladder slips away from the wall at a rate of 0.5 meters per second, at what rate is the top of the ladder sliding towards the ground when the foot of the ladder is 1 meter away from the wall?

We have \( x^2 + y^2 = 4^2 \). Thus, \( 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \). Solve for \( \frac{dx}{dt} \) and determine \( y \) when \( x = 1 \).

\[
\left. \frac{dy}{dt} \right|_{x=1} = - \frac{x \left( \frac{dx}{dt} \right)}{y} \bigg|_{x=1} = - \frac{1 (0.5)}{\sqrt{4^2 - 1}}
\]
\[
= - \frac{1}{2\sqrt{15}} \approx -0.129 \text{ meters per second}
\]
17. Starting with $x_0 = 1$, use Newton’s method to find the second approximation, $x_2$, to the root of $x^3 - 2 = 0$.

Newton’s method takes an initial guess $x_0$ and computes $x_0 - \frac{f(x_0)}{f'(x_0)}$, and sets this equal to $x_1$.

For our function we have $f(x) = x^3 - 2$, $f'(x) = 3x^2$. Let $F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 2}{3x^2} = \frac{2x}{3} + \frac{2}{3x^2}$.

Then we have

\[
\begin{align*}
  x_1 &= F(1) = \frac{4}{3} \\
  x_2 &= F(x_1) = F\left(\frac{4}{3}\right) = \frac{91}{72} \approx 1.264
\end{align*}
\]

18. Let $f(x) = x - x^{1/3}$. Is this function one-to-one on the interval $[-100, 100]$? An explanation which consists solely of "..., because my graphing calculator shows..." will not be worth much.

This function is not one-to-one on the interval $[-100, 100]$ as $f(-1) = f(0) = f(1) = 0$. The following is a plot of $f(x)$, but not over the entire interval $[-100, 100]$. 

![Graph of f(x)](image-url)