1. (10) Define the following:

(a) \( \lim_{x \to 3^+} f(x) = 11 \)

Means that for any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if

\[
0 < x - 3 < \delta, \quad |f(x) - 11| < \epsilon
\]

(b) \( x_0 \) is the location of a local minimum of a function \( f(x) \).

To say that \( x_0 \) is the location of a local minimum of \( f(x) \) means that there is an open interval \((a, b)\) containing the point \( x_0 \), that is \( a < x_0 < b \), such that for any \( x \) in \((a, b)\) we have \( f(x_0) \leq f(x) \).

2. (10) State the Mean Value Theorem, and give a geometrical interpretation of it.

Let \( f(x) \) be continuous on the interval \([a, b]\) and differentiable on the interval \((a, b)\). Then there is a point \( c \) in the open interval \((a, b)\) such that

\[
f(b) - f(a) = f'(c)(b - a)
\]

A geometrical interpretation of this is, there is a point \( c \) between \( a \) and \( b \), such that the tangent line to the graph of \( f \) at the point \((c, f(c))\) is parallel to the secant line connecting the two points \((a, f(a))\) and \((b, f(b))\).

3. (15) Compute the derivatives of the following functions.

(a) \( \frac{d}{dx} \left( \frac{2x - 1}{\sin x} \right) = \)

\[
\frac{d}{dx} \left( \frac{2x - 1}{\sin x} \right) = \frac{2 \sin x - (2x - 1) \cos x}{\sin^2 x}
\]

(b) \( \frac{d}{dx} (\cos (\tan 2x)) = \)

\[
\frac{d}{dx} (\cos (\tan 2x)) = -2 \sin (\tan 2x) \sec^2 (2x)
\]

(c) \( \frac{d}{dt} \left( (t^2 - 5t)^9, \sec t \right) = \)

\[
\frac{d}{dt} \left( (t^2 - 5t)^9, \sec t \right) = \left( 9 (t^2 - 5t)^8 (2t - 5), \sec t \tan t \right)
\]
4. (10) Find the tangent line to the curve \(x^2 (3y + 2)^4 + x (y + 1) = 4\) at the point \((2, -1)\).

Differentiate the equation.

\[
2x (3y + 2)^4 + 4x^2 (3y + 2)^3 \left(3y'\right) + (y + 1) + xy' = 0 \text{ now solve for } y'
\]

\[
y' \bigg|_{(2,-1)} = -\frac{2x (3y + 2)^4 + (y + 1)}{12x^2 (3y + 2)^3 + x} \bigg|_{(2,-1)} = -\frac{4}{12 \cdot 4 \cdot (-1) + 2}
\]

\[
= -\frac{4}{-46} = \frac{2}{23}
\]

Thus, an equation for the tangent line is

\[
y + 1 = \frac{2}{23} (x - 2)
\]

5. (10) Suppose that \(f\) is a differentiable function, and that the line \(y = 8 - 3x\) is the tangent line approximation to \(f\) at \(x = 4\).

(a) What must \(f(4)\) and \(f'(4)\) equal?

\[
f(4) = 8 - 12 = -4
\]

\[
f'(4) = -3
\]

(b) What is a reasonable approximation to \(f(3.95)\)?

\[
f(3.95) \approx 8 - 3(3.95) = -3.85
\]

6. (15) Newton’s method is a powerful algorithm used to solve equations of the form \(f(x) = 0\).

(a) Derive Newton’s method.

The idea is to start at the initial guess \(x_1\), and find where the tangent line to the graph of \(f\) at the point \((x_1, f(x_1))\) crosses the \(x\)-axis. The equation of the tangent line is

\[
y = x_1 + f'(x_1)(x - x_1)
\]

If \(x_2\) denotes the \(x\) coordinate of where the tangent line crosses the \(x\)-axis, then \(x_2\) must satisfy the equation \(0 = x_1 + f'(x_1) (x_2 - x_1)\). Solving for \(x_2\) we have

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
\]

(b) Using Newton’s method to solve the equation \(x^4 - 3 = 0\), if \(x_1 = 1\) is a guess to the positive real solution, what would \(x_2\) equal.

Set \(f(x) = x^4 - 3\), then \(f'(x) = 4x^3\). and the equation for Newton’s method takes the form

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
= x_n - \frac{(x_n)^4 - 3}{4(x_n)^3} = x_n - \frac{x_n}{4} + \frac{3}{4x_n^3}
\]

\[
= \frac{3}{4} \left( x_n + \frac{1}{x_n^3} \right)
\]

So if \(x_1 = 1\), then \(x_2 = (3/4)(1 + 1) = 3/2\).
7. (20) Suppose \( f(x) = \frac{x^4}{4} - \frac{x^3}{6} - \frac{x^2}{2} + \frac{3}{2}x, -\infty < x < \infty. \)

(a) Determine where \( f \) is increasing and decreasing. Hint: \( f' \) has three roots. They are \(-3/2, 1, \) and \(1.
\[
f'(x) = x^3 - \frac{x^2}{2} - 2x + \frac{3}{2} = (x + \frac{3}{2})(x - 1)^2
\]
Thus, the derivative is negative if \( x < -3/2 \), and positive if \( x > -3/2 \). So \( f \) is decreasing on \((-\infty, -3/2)\) and increasing on \((-3/2, \infty)\).

(b) Determine where \( f \) is concave up and concave down.
\[
f''(x) = 3x^2 - x - 2 = (3x + 2)(x - 1)
\]
Thus, the second derivative is positive for \( x < -2/3 \), negative for \(-2/3 < x < 1 \), and positive for \( x > 1 \). Hence \( f \) is concave up for \( x < -2/3 \), concave down for \(-2/3 < x < 1 \), and concave up for \( x > 1 \).

(c) Find all local extrema.
A local extrema can occur only at a critical point of \( f \). Since \( f \) is differentiable everywhere, local extrema can occur only at places where the derivative is zero. That is, at \( x = -3/2 \) or \( x = 1 \). At \(-3/2 \), the second derivative is positive so we have a local minimum at that point. At \( x = 1 \), the function does not have a local extremum. First the second derivative is zero there, which is a warning to be careful, and when we examine the values of the first derivative on either side of \( x = 1 \), we see that these values are positive. So for \( x < 1 \) we would have \( f(x) < f(1) \), and for \( x > 1 \), we would have \( f(1) < f(x) \). It is understood of course that these inequalities hold for \( x \) close to 1, and not necessarily for all \( x \).
In conclusion, we have one local extremum, a local minimum at \( x = -3/2 \).

(d) Find all points of inflection.
The second derivative of \( f \) equals \((3x + 2)(x - 1)\). Clearly the second derivative has sign changes at \( x = -2/3 \), and \( x = 1 \). So the only points of inflection occur at \((-2/3, -109/81)\) and \((1, 7/12)\).

(e) Sketch a plot of \( f \).
8. (10) A cylindrical can without a top is made to contain 1000\(\pi\) cm\(^3\) of liquid. Find the values for the radius and height that will minimize the amount of material needed to make the can.

Let \(r\) and \(h\) denote the radius and height of the can respectively. Then the volume of the can is \(V = \pi r^2 h\), and its surface area is \(S = \pi r^2 + 2\pi rh\). Since the volume equals 1000\(\pi\) we have

\[
1000\pi = \pi r^2 h
\]

\[
h = \frac{1000}{r^2}
\]

Thus,

\[
S = \pi r^2 + 2\pi r \left( \frac{1000}{r^2} \right) = \pi r^2 + \frac{2000\pi}{r}
\]

As \(r\) approaches zero from above the surface area becomes arbitrarily large as it also does when \(r\) goes to plus infinity. So for some finite non-zero value of \(r\), the surface area will be have a global minimum, and since an interior point for which a function has a global minimum is also a local minimum we need to find those \(r\)'s for which \(S' = 0\).

\[
\frac{dS}{dr} = 2\pi r - \frac{2000\pi}{r^2} = 0 \text{ implies}
\]

\[
r^3 = 1000 = 10^3
\]

\[
r = 10 \text{ and } h = \frac{1000}{10^2} = 10
\]