A yes or no answer or an answer with no justification will not be acceptable. Remember to write neatly and clearly and in sentences.

1. (20) Define the following terms and/or symbols and give an example of each. No example no credit.

(a) contrapositive

**Ans:** If we have an implication \( P \rightarrow Q \) its contrapositive is the implication \( \neg Q \rightarrow \neg P \). As an example the contrapositive of the statement \( \neg Q \rightarrow \neg P \) is the statement \( \neg P \rightarrow \neg Q \). If 2 is less than 3 then 3 is less than 4. If 3 is greater than or equal to 4 then 2 is greater than or equal to 3.

(b) tautology

**Ans:** A tautology is a statement that is always true regardless of the truth values of its component statements. An example of a tautology is \( P \lor \neg P \).

(c) \( A - B \) where \( A \) and \( B \) are two sets

**Ans:** \( A - B \) is defined as the set of elements that are in \( A \) and not in \( B \). If we let \( A = \{1, 2, 3\} \) and \( B = \{1, 2\} \) then \( A - B = \{3\} \).

(d) \( \bigcup_{\gamma \in \Gamma} A_\gamma \)

**Ans:** In this notation \( \Gamma \) is referred to as an indexing set. That is each member \( \gamma \) of \( \Gamma \) has a set associated with it \( A_\gamma \) and the notation \( \bigcup_{\gamma \in \Gamma} A_\gamma \) represents the set which consists of all elements that are in at least one of the sets \( A_\gamma \). For example let \( \Gamma = \{1, 2, 3\} \), \( A_1 = \{2\} \), \( A_2 = \{3\} \), and \( A_3 = \{4\} \). Then \( \bigcup_{\gamma \in \Gamma} A_\gamma = \{2, 3, 4\} \).

2. (20) Is the following a tautology, a contradiction, or neither?

\[
[(P \rightarrow Q) \land (Q \rightarrow P)] \rightarrow (P \land Q) \lor (\neg P \land \neg Q)
\]

**Ans:** This is a tautology. One way to see this is to construct a truth table for this statement and verify that it is always true.
3. (20) Prove or disprove the following:

\[(A - B) - C = A - (B - C),\]

where \(A, B,\) and \(C\) are sets.

**Ans:** This is not a true statement. To see how to construct a counter example notice that the LHS can contain no elements in the set \(C\) while that is not true for the RHS. Thus let \(A = \{1\}, B = \{2\},\) and \(C = \{1\}.\) Then \((A - B) - C = \emptyset \) and \(A - (B - C) = \{1\}.\)

4. (15) Assign a grade of A (correct)\(\Gamma\) C (partially correct)\(\Gamma\) F (wrong) to the following “proof”. Justify all grades other than an A.

**Claim:** Let \(N\) denote the natural numbers. If \(X = \{x \in N : x^2 < 14\}\) and \(Y = \{1, 2, 3\}\) then \(X = Y.\)

**Proof:** Since \(1^2 = 1 < 14, 2^2 = 4 < 14,\) and \(3^2 = 9 < 14\) we have \(X = Y.\)

**Ans:** I would give a grade of C. This proof is not complete. It does correctly show that the set \(Y\) is a subset of the set \(X.\) However there is no indication that the author of this ‘proof’ ever worried about whether or not \(X\) is a subset of \(Y.\)

5. (15) Let \(A = \{1, 2, 3, 4\}.\) Let \(\mathcal{P}(A)\) denote the power set of \(A.\) Which of the following are true statements? Be sure to explain.

(a) \(1 \in \mathcal{P}(A)\)

**Ans:** This is not true. The element 1 is a member of \(A\) but it is not a subset of \(A.\) Since \(\mathcal{P}(A)\) is defined to be the collection of all subsets of \(A\) 1 does not belong to \(\mathcal{P}(A).\)

(b) \(\{1, 2\} \subseteq \mathcal{P}(A)\)

**Ans:** This is a false statement. Each of the elements 1 and 2 belong to \(A.\) Thus \(\{1, 2\}\) is a subset of \(A\) and belongs to \(\mathcal{P}(A).\) Thus we should write \(\{1, 2\} \in \mathcal{P}(A).\) When we write \(\{1, 2\} \subseteq \mathcal{P}(A)\) we are saying that \(\{1, 2\}\) is a subset of \(\mathcal{P}(A).\) This means we are saying that both of the elements 1 and 2 belong to \(\mathcal{P}(A)\) and as we saw in part (a) this is not true.
6. (10) For each $\alpha \in \mathcal{A}$ let $A_\alpha$ be a set. In class we saw that the following set inclusion is true:

$$
\bigcup_{\alpha, \beta \in \mathcal{A}} (A_\alpha - A_\beta) \subseteq \bigcup_{\alpha \in \mathcal{A}} A_\alpha - \bigcap_{\alpha \in \mathcal{A}} A_\alpha.
$$

Is it possible for the left hand side to be a proper subset of the right hand side or must the two sides always equal each other?

**Ans:** It is not possible for the LHS to be a proper subset of the RHS. The two sets are equal. To see this suppose $x$ belongs to the RHS. Then there are $\alpha_0$ and $\beta_0$ such that $x \in A_{\alpha_0}$ and $x \notin A_{\beta_0}$. Thus we have $x \in A_{\alpha_0} - A_{\beta_0}$. Hence $x$ belongs to the LHS also and we have shown the RHS is a subset of the left hand side. Since we already know that the LHS is a subset of the RHS the two sets are equal.