Be sure to show and/or explain your work, bald answers are not worth much.

1. (20) Let $\Gamma(t) = (\cos t, \sin t, 8 - t/2)$ for $0 \leq t \leq 8$, be the path of a fly in our classroom.

   (a) Where will the fly be located when $t = 8$?
   
   $\Gamma(8) = (\cos 8, \sin 8, 4) \approx (-0.145, 0.989, 4)$

   (b) What is the fly’s velocity and speed at $t = \frac{\pi}{4}$?
   
   $\Gamma'(t) = \left(-\sin t, \cos t, \frac{-1}{2}\right)$. Thus, $\Gamma'(\frac{\pi}{4}) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{2}\right)$
   
   and the speed equals $||\Gamma'(\frac{\pi}{4})|| = \frac{\sqrt{5}}{2}$.

   (c) Find the projection of the fly’s velocity vector at $t = \frac{\pi}{4}$ onto the vector $(1, 1, 1)$. The projection equals

   \[
   \frac{\Gamma'(\frac{\pi}{4}) \cdot (1, 1, 1)}{||(1, 1, 1)||^2} (1, 1, 1) = \frac{-1}{6} (1, 1, 1)
   \]

   (d) How far does the fly fly?

   \[
   \int_0^8 ||\Gamma'(t)|| \, dt = \int_0^8 \sqrt{(-\sin t)^2 + \cos^2 t + 1/4} \, dt
   \]

   \[
   = \int_0^8 \frac{\sqrt{5}}{2} \, dt = 4\sqrt{5}.
   \]

2. (20) Let $S$ be the surface which is the graph of the function $f(x, y) = x^3 - xy + xy^2$. That is, $S = \{(x, y, z) : z = f(x, y)\}$.

   (a) What is the domain of $f$?

   domain equals all of $\mathbb{R}^2$

   (b) What is the range of $f$?

   range equals all real numbers
(c) Find the equation for the tangent plane to $S$ at the point $(1, 2, 3)$.

Let $T(x, y)$ denote the equation of the plane.

\[
T(x, y) = f(1, 2) + (x - 1) \frac{\partial f}{\partial x}(1, 2) + (y - 2) \frac{\partial f}{\partial y}(1, 2) \\
= 3 + 5(x - 1) + 3(y - 2) \\
= 5x + 3y - 8.
\]

3. (20) Let

\[
f(x, y, z) = xy - \sin(yz),
\]

\[
\Gamma(t) = (e^t - 2, t^2 - 1, \ln t)
\]

\[
g(t) = f(\Gamma(t))
\]

Find the rate of change of $g(t)$ when $t = 1$.

\[
\Gamma(1) = (e - 2, 0, 0) \text{ and } \Gamma'(t) = \left(e^t, 2t, \frac{1}{t}\right)
\]

\[
\nabla f = (y, x - z \cos yz, -y \cos yz) \text{ Thus, at } (e - 2, 0, 0)
\]

\[
\nabla f = (0, e - 2, 0)
\]

\[
\frac{dg}{dt} = \nabla f \cdot \Gamma'(1) = (0, e - 2, 0) \cdot (e, 2, 1) = 2(e - 2).
\]

4. (20) Let $f(x, y) = \frac{xy^2}{x^2 + y^4}$.

(a) \[\lim_{(x, y) \to (2, 1)} f(x, y) = ?\]

\[\lim_{(x, y) \to (2, 1)} f(x, y) = \frac{2}{5}\]

(b) Define, in terms of $\epsilon$ and $\delta$, what \[\lim_{(x, y) \to (2, 1)} f(x, y) = (\text{your answer from a.})\] means.

for any $\epsilon > 0$ there exists a $\delta > 0$ such that

if $0 < \| (x, y) - (2, 1) \| < \delta$, then $\left\| \frac{xy^2}{x^2 + y^4} - \frac{2}{5} \right\| < \epsilon$

(c) \[\lim_{(x, y) \to (0, 0)} f(x, y) = ?\]

The limit does not exist. If we let $(x, y) \to (0, 0)$ along the line $x = y$, then the limiting value is zero. However, if we let $(x, y) \to (0, 0)$ along the curve $x = y^2$ then the limiting value is $1/2$. 2
5. (20) The function

\[ T(x, y, z) = x e^{-y} + z e^{-x} + y z, \text{ for } x^2 + y^2 + z^2 \leq 4, \]

gives the temperature at each point of a sphere of radius 2 centered at the origin.

(a) If you are at the origin, which direction should you head if you wanted to get cooler quickly?

\[ \nabla T = (e^{-y} - z e^{-x}, -x e^{-y} + z, e^{-x} + y) \]

\[ \nabla T(0, 0, 0) = (1, 0, 1). \]

Thus, to cool off the quickest, head in the direction \(- (1, 0, 1)\).

(b) The gradient of \( T \) is non-zero. What can we conclude from this about where the maximum and minimum temperatures occur?

That the extreme values occur on the boundary \( x^2 + y^2 + z^2 = 4 \), since there are no local extrema.

(c) Explain how you would go about finding the locations of the global maximum and global minimum of the function \( T \). Be sure to explicitly and carefully describe what you would do.

Use Lagrange multipliers. Set \( g(x, y, z) = x^2 + y^2 + z^2 \) and then solve the system of equations

\[ \nabla T + \lambda \nabla g = \vec{0} \]

\[ g = 4. \]