1. (30) Let \( R = \{(x, y) : x^2 + y^2 \leq 1, \text{ and } 0 \leq y\} \). Let \( F(x, y) = \left( x + \frac{y^2}{2}, 2x + xy \right) \).

Hint: remember Green’s theorem.

(a) Find the integral of the tangential component of \( F \) along \( \partial R \), where \( \partial R \) is traversed in a counterclockwise direction.

\[
\int_{\partial R} F \cdot dR = \int \int_{R} [(2 + y) - y] \, dA = \int \int_{R} 2 \, dA = 2 \text{(area of half circle)} = 2 \frac{\pi}{2} = \pi
\]

(b) Find the integral of the outward normal component of \( F \) along \( \partial R \), where \( \partial R \) is traversed in a counterclockwise direction.

The outward normal component of \( F \) is given by \( F \cdot \frac{(y', -x')}{\sqrt{(x')^2 + (y')^2}} \).

The line integral of this component around the curve equals

\[
\int_{\partial R} F \cdot \frac{(y', -x')}{\sqrt{(x')^2 + (y')^2}} \sqrt{(x')^2 + (y')^2} \, dt = \int_{\partial R} F \cdot (y', -x') \, dt
\]

\[
= \int_{\partial R} \left( x + \frac{y^2}{2} \right) dy - (2x + xy) \, dx
\]

\[
= \int \int_{R} (1 + x) \, dA \text{(this is Green’s theorem)}
\]

\[
= \int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} (1 + x) \, dy
\]

\[
= \int_{-1}^{1} (1 + x) \sqrt{1 - x^2} \, dx
\]

\[
= \frac{\pi}{2} \text{.}
\]

2. (35) A force field \( F \) is said to be conservative if there is a function \( \phi \) such that \( \nabla \phi = F \). That is, \( F \) is the gradient of a function.
(a) Let $\Gamma_1$ and $\Gamma_2$ be two different paths connecting the points $(a_1, b_1)$ and $(a_2, b_2)$ in the given order. That is both $\Gamma_1$ and $\Gamma_2$ start at $(a_1, b_1)$ and end at $(a_2, b_2)$. Explain why, assuming $F$ is conservative, that

$$\int_{\Gamma_1} F \cdot d\Gamma = \int_{\Gamma_2} F \cdot d\Gamma$$

Since $F = \nabla \phi$, we can write the line integral as follows (assume $\Gamma(0) = (a_1, b_1)$ and $\Gamma(1) = (a_2, b_2)$)

$$\int_{\Gamma} F \cdot d\Gamma = \int_{\Gamma} \left( \frac{\partial \phi}{\partial x} \right) dx + \left( \frac{\partial \phi}{\partial y} \right) dy$$

$$= \int_{0}^{1} \left[ \left( \frac{\partial \phi}{\partial x} \right) \frac{dx}{dt} + \left( \frac{\partial \phi}{\partial y} \right) \frac{dy}{dt} \right] dt$$

$$= \int_{0}^{1} \left[ \frac{d}{dt} \phi(\Gamma(t)) \right] dt$$

$$= \phi(\Gamma(1)) - \phi(\Gamma(0))$$

$$= \phi(a_2, b_2) - \phi(a_1, b_1)$$

Thus, the value of the line integral does not depend upon the path chosen, but only upon the values of the potential function at the endpoints of the path.

(b) Let $F(x, y, z) = (f_1, f_2, f_3)$, where each $f_i$ is a function of $x, y,$ and $z$. What are the necessary conditions which $F$ must satisfy for it to be conservative. If $F$ satisfies these necessary conditions, must it be conservative? Explanations required.

The necessary conditions are

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}.$$

If $F$ satisfies these conditions, and if the region over which $F$ is defined is simple connected, then $F$ is a conservative force field. If the region is not simply connected, then $F$ need not be conservative over its domain.

(c) Let $F(x, y, z) = (2xyz, x^2z + \sin z, x^2y + y \cos z + 2)$. Find $\phi$ such that $\nabla \phi = F$.

$$\phi(x, y, z) = x^2yz + y \sin z + 2z + c,$$ where $c$ is any constant.
(d) Let $F(x, y, z) = (2xyz, x^2y + y \sin z + 2)$. Let $\Gamma(t) = (t^2, \sin(t) \cos t^2, \ln(1 + t))$, for $0 \leq t \leq \pi$. Compute $\int_{\Gamma} F \cdot d\Gamma$.

Since $F$ is conservative (see part c.), we know that the value of the line integral of the tangential component of $F$ equals $\phi$ (end-point) – $\phi$ (start-point). The starting and ending points of $\Gamma$ are found below

$$\Gamma(\pi) = (\pi^2, 0, \ln(1 + \pi)) \quad \Gamma(0) = (0, 0, 0)$$

Thus, we have

$$\int_{\Gamma} F \cdot d\Gamma = \phi(\pi^2, 0, \ln(1 + \pi)) – \phi(0, 0, 0)$$

$$= 2 \ln(1 + \pi) – 0$$

$$= 2 \ln(1 + \pi)$$

3. (35) Let $S$ be that part of the wall of the cylinder $x^2 + y^2 = 16$ which lies between the planes $z = 0$ and $z = 2$.

(a) Find a parametric representation of $S$.

$$\vec{r}(\theta, z) = (4 \cos \theta, 4 \sin \theta, z) \quad \text{for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq z \leq 2.$$ 

(b) Set up and evaluate an integral which gives the surface area of $S$.

$$\frac{\partial \vec{r}}{\partial \theta} = (-4 \sin \theta, 4 \cos \theta, 0), \quad \frac{\partial \vec{r}}{\partial z} = (0, 0, 1),$$

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = (-4 \sin \theta, 4 \cos \theta, 0) \times (0, 0, 1)$$

$$= (4 \cos \theta, 4 \sin \theta, 0)$$

The integral which gives the surface area is

$$\int \int_{S} dS = \int \int_{R} \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} \right\| dA$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{2} \left\| (4 \cos \theta, 4 \sin \theta, 0) \right\| dz$$

$$= 4 \int_{0}^{2\pi} d\theta \int_{0}^{2} dz$$

$$= 16\pi.$$
(c) Evaluate \( \iint_S (x^2 + y^2) \, dS. \)

\[
\iint_S (x^2 + y^2) \, dS = \int_0^{2\pi} d\theta \int_0^2 \left( (4 \cos \theta)^2 (4 \sin \theta) + z^2 \right) \left\| (4 \cos \theta, 4 \sin \theta, 0) \right\| \, dz
\]

\[
= 4 \int_0^{2\pi} d\theta \int_0^2 \left( (4 \cos \theta)^2 (4 \sin \theta) + z^2 \right) \, dz
\]

\[
= 4 \int_0^{2\pi} d\theta \int_0^2 \left[ 64 \cos^2 \theta \sin \theta + z^2 \right] \, dz
\]

\[
= 4 \left[ 0 \cdot \frac{16\pi}{3} \right] \text{ Note that } \cos^2 \theta \text{ is even about } \pi \text{ and that sine is odd about } \pi. \text{ Hence their product is odd about } \pi.
\]

\[
= \frac{64\pi}{3}
\]